2^L - SYMBOL 8-DIMENSIONAL CONSTELLATIONS

Kenneth Cartwright
College of the Bahamas
Technology Division
P.O.Box N-4912
Nassau, Bahamas.

Abstract

By utilizing four consecutive symbol periods at a ime, eight-dimensional constellations can be generated. ere, such 2^L -symbol constellations (herein called 8D-L) re proposed, where L is any integer $\geq$ 8. Actually, two onstellations are given for L = 8: one which requires an ptimum detector, 8D-8B, and the other, 8D-8A, which can e detected by the general detector provided for 8D-L. It s demonstrated that 8D-8B provides 0.37 dB improvement as far as Signal-to-Noise Ratio is concerned) over 8D-8A hich, in turn, provides 1.78 dB over the corresponding PSK system. Furthermore, 8D-12 is better than 8-QAM by .17 dB, whereas 8D-16 achieves 2.55 dB improvement over 6-QAM.

Introduction

Gersho and Lawrence [1] have presented eight-imensional constellations by utilizing four consecutive ymbol periods at a time. Each symbol period is used to ransmit a two-dimensional AM-PM constellation. They laimed that their eight-dimensional 2^{16} - symbol onstellation S_8 can achieve 2.4 dB improvement in Signal-o-Noise Ratio (SNR) performance compared with 16-QAM ver four symbol periods. However, a disadvantage of the euristic method of Gersho and Lawrence is that it is not pparent how other constellation sizes may be ccommodated. For example, how can their scheme be made o transmit 2^L symbols, L $\geq$ 8, over four symbol periods. urthermore, the detection process for S_8 is somewhat omplex. Even though the detector must find the minimum f only twenty distances each block (four symbols) eriod, the method used depends upon to which of four ubgroups the received symbol belongs. This fact omplicates the detection process.

In this paper, the author proposes 2^L - symbol 8-imensional constellations, hereafter called 8D-L, which equire relatively simple detectors. These detectors equire finding the minimum of only sixteen distances very block period. In addition, there are low-level uantizers in each co-ordinate of the received vector. In eneral, the level of quantization depends upon L. owever, for L=8, 12, and 16, there are only three evels. The use of quantizers replaces the need of ermuting the co-ordinates of the received vector and nverse permuting the selected basic point, as required y S_8. See [1] for details.

Actually, the author proposes two different onstellations for L = 8: the one mentioned above, ereafter called 8D-8A, and another one which will be alled 8D-8B. It will be seen that 8D-8B has 0.37 dB dvantage over 8D-8A, at the expense of a slightly more omplex detector. In fact, 8D-8B requires finding the inimum of twenty-six distances every block period.

Description of the 8D-L Constellation

In this section, the eight-dimensional 8D-L onstellation will be described. To begin, let any ransmitted symbol p_i be written as $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$, where a_j, b_j, j=1, 2, 3, 4, s the two-dimensional AM-PM constellation of the j^{th} ymbol period. Then, the 8D-L constellation will be given y $\{p_i\}$, for i=1 to 2^L, L $\geq$ 8. The values of a_j, b_j will be ntegers, which depend upon L. In addition, p_i can be any f the sixteen basic points listed in Table 1. Note that n Table 1 E denotes an even integer, whereas O denotes an odd integer. Notice also from Table 1 that any two basic points differ in at least four co-ordinates. This means that the Minimum Square Euclidean Distance (MSED) will be four, the same as it is with regular QAM.

To form 8D-L, M = 2^L points are generated from Table 1, so that the average energy of the constellation P is minimized, where

$$P = \frac{1}{M} \sum_{i=1}^{M} \|p_i\|^2. \tag{1}$$

Table 1. Basic points for the 8D-L constellation. E denotes an even integer. O denotes an odd integer.

Basic Point	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4
1	O	O	O	O	E	E	E	E
2	O	O	E	E	O	O	E	E
3	O	O	E	E	E	E	O	O
4	E	E	E	E	O	O	O	O
5	E	E	O	O	E	E	O	O
6	E	E	O	O	O	O	E	E
7	O	E	O	E	E	O	E	O
8	O	E	E	O	O	E	E	O
9	O	E	E	O	E	O	O	E
10	E	O	E	O	O	E	O	E
11	E	O	O	E	E	O	O	E
12	E	O	O	E	O	E	E	O
13	E	O	E	O	E	O	E	O
14	O	E	O	E	O	E	O	E
15	O	O	O	O	O	O	O	O
16	E	E	E	E	E	E	E	E

8D-8 Constellation

For 8D-8, let E be zero, and O be -1, or 1, for the first fourteen basic points of Table 1. This then gives 14x16=224 unique points. The sixteen follows because there are four odd co-ordinates and each of these can have two values (-1 or 1). Therefore, there are 2^4 unique symbols for each of the first fourteen points of Table 1. Furthermore, 16 additional points can be derived from Basic Point #16, if one E is allowed to be -2 or 2, and the other co-ordinates are made equal to zero. Finally, for 8D-8A, the remaining 16 points can be had from Basic Point #15. In fact, a_1, b_1, a_2, and b_2 are set equal to 1, and a_3, b_3, a_4, b_4 can be -1 or 1. However, for 8D-8B, the remaining 16 points are the sign variations of [0 0 0 0 0 0 2 2], [0 0 0 0 2 2 0 0], [0 0 2 2 0 0 0 0], and [2 2 0 0 0 0 0 0]. Table 2 summarizes the makeup of 8D-8A, and 8D-8B.

8D-12 and 8D-16 Constellations

Tables 3 and 4 show the makeup of 8D-12 and 8D-16, respectively. A word will now be given on how these points were selected. Inspection of these tables and Table 1 shows that there are three basic kinds of points-those whose co-ordinates are (i) all odd, (ii) all even, and (iii) those which have half odd and half even. These will be called odd points, even points, and mixed points, respectively.

For a given energy level, say 12, the possible odd, even, and mixed points are listed. It should be obvious that there are no odd points in 8D, whose energy is 12. The only even points are [0 0 0 0 0 2 2 2], and all of its permutations and sign variations. For the mixed

Table 2. Makeup of the 8D-8A and 8D-8B constellations.

Type of point	Energy	No. of points Due to sign variations	Due to permutation among even	odd	Due to type of basic point	Total No. of points
11110000 (Both)	4	x16			x14	224
00000002 (Both)	4	x2	x8			16
11111111*(8D-8A)	8	x16				16
00000022 (8D-8B)	8	x4				4
00002200 (8D-8B)	8	x4				4
00220000 (8D-8B)	8	x4				4
22000000 (8D-8B)	8	x4				4

* The first four co-ordinates are always 1.

Table 3. Makeup of the 8D-12 constellation.

11110000	4	x16			x14	224
00000002	4	x2	x8			16
11111111	8	x256				256
00000022	8	x4	x28			112
00021111	8	x32	x4		x14	1792
00000222*	12	x8	x20			160
00221111	12	x64	x6			1536

* a_1, b_1 are always zero, so that only the last six co-ordinates are permutated.

Table 4. Makeup of the 8D-16 constellation.

11110000*	4	x16			x10	160
00000002	4	x2	x8			16
11111111	8	x256				256
00000022	8	x4	x28			112
00021111	8	x32	x4		x14	1792
00000222	12	x8	x56			448
00221111	12	x64	x6		x14	5376
00001113	12	x16		x4	x14	896
02221111	16	x128	x4		x14	7168
00021113	16	x32	x4	x4	x14	7168
00002222	16	x16	x70			1120
11111113	16	x256		x8		2048
00001133	20	x16		x6	x14	1344
00221113	20	x64	x6	x4	x14	21504
22221111	20	x256			x14	3584
00022222	20	x32	x56			1792
00021133	24	x32	x4	x6	x14	10752

*The last four basic point types of Table 1 were omitted.

Table 5. Necessary constants used to compute P_{es} with (2) and (4), and gain of 8D-L over the corresponding M-QAM system.

Constellation	Energy	Optimum K from (4)	Gain in P (dB)	Error Co-efficient Loss (dB)	Overall Gain (dB)
8D-8A	4.25	58.5	2.747	0.981	1.766
8D-8B	4.25	53.47	2.747	0.548	2.199
8D-12	9.42	117.1	4.062	0.917	3.145
8D-16	18.33	NA	3.389	0.864	2.525
QPSK	8	8			
8-QAM	24	10			
16-QAM	40	12			

points, there are two cases-(i) the odd co-ordinates have energy level 4 and the even co-ordinates have energy level 8, and (ii) the odd co-ordinates have energy level 12and the even have energy level 0. Points in case (i) are given by [0 0 2 2 1 1 1 1], and all of its permutations, e.g. [0 2 0 2 1 1 1 1], and sign variations, whereas, for case (ii), [0 0 0 0 1 1 1 3], and all of its permutations, e.g. [0 0 0 0 3 1 1 1], and sign variations are allowed points. *Notice that by permutations, it is meant permutations among the even co-ordinates, or among the odd co-ordinates, and not permutations among all eight co-ordinates.* For example, [0 3 1 1 1 0 0 0] is not allowed. Notice also that a mixed point can be used 14 times just by changing the order of the even and odd co-ordinates, as Table 1 allows.

At all energy levels, the above procedure is followed, beginning with the lowest energy first, so that P is minimized. Notice that energy levels are in steps of 4. This is a consequence of the allowed points of Table 1.

Detecting 8D-L

To illustrate the method of detection, the two-dimensional M-symbol constellation of Fig. 1 will be used. Let r_2 be the two-dimensional received vector which is of course the transmitted symbol s_2 corrupted with AWGN of zero mean and variance σ^2 in each co-ordinate. Notice that the constellation can be thought of as being made up of adjacent rectangles, called stacking rectangles. These rectangles have one point enclosed, except the one at the origin. Notice also that some of the rectangles have some vertices missing. The detection procedure to be descibed assumes that there are no missing symbols. This means that the detection procedure is not optimum, but fortunately, as will be demonstrated, there is very little degradation. In fact, the missing symbols only cause the error coefficient to increase somewhat (See (4)). It is well known that the probability of symbol error P_{es} is not a strong function of the error coefficient. According to Forney [2], for every doubling of the error coefficient, SNR decreases by 0.2 dB.

Referring to Fig. 1, suppose that r_2 is in rectangle 4. Then the transmitted symbol is most likely to have been a vertex or the center point of rectangle 4. Therefore, only the Euclidean distances between r_2 and the vertices and center point of rectangle 4 need to be found. The symbol of this group of five which minimizes the computed distances is declared to be the transmitted symbol. Thus, the detection problem has been reduced from finding the minimum of M distances to finding the minimum of just five distances. However, one problem still remains - the five candidate symbols depend upon the rectangle in which r_2 is located. This complicates the detection procedure. However, the candidate symbols need not change if distances to $r_2 - c_2$ are computed instead, where c_2 is the center of the received rectangle. This follows since $r_2 - c_2$ translates the received rectangle to the origin. Therefore, the five candidate symbols are now just the vertices and center point of the rectangle at the origin. Note that c_2 can be found quite easily with low-level quantization in each co-ordinate.

Let w be the symbol which minimizes the distance between the received translated vector $d_2 = r_2 - c_2$ and the five candidate symbols. Then, the detected symbol is given by $\hat{s}_2 = w + c_2$.

Actually, the detection process can be further simplified by making use of the symmetry of the five candidate symbols, so that finding w is quicker. Notice that if the absolute value of each co-ordinate of the received translated vector is taken (denote this as $|d_2|$ $= |r_2 - c_2|$), then the four vertices become just one point. Therefore, there are now only two candidate symbols - the vertex in the first quadrant and the origin. If v is the symbol which minimizes the distance between $|d_2|$ and the two candidate symbols, then w = v.*sgn(d_2), where sgn(d_2) is d_2 with a sgn operation in each co-ordinate, and .* is element-by-element multiplication.

The detection process can be summarized and generalized to n-dimensions as follows:

Step 1. Find the rectangle (n-dimensional hypercube) in which the received vector is located. This is done by using quantizers in each co-ordinate.

Step 2. Translate the received rectangle (hypercube) to the origin. This is simply achieved by subtracting the output of the quantizers from the received vector.

Step 3. Find the symbol which minimizes the Euclidean distance between the translated received vector and the candidate symbols associated with the rectangle (hypercube) at the origin.

Step 4. Add this detected optimum symbol to the center of the received rectangle (hypercube), so that the optimum symbol will be translated back to its proper location.

Step 5. Step 3 can be simplified as follows:

(i) Map (many-to-one) the received translated vector onto the first quadrant (positive sector in n-space, i.e. space where the vectors have co-ordinates that are non-negative). This is done simply by finding the absolute value of each co-ordinate of the received translated vector.

(ii) Find the symbol which minimizes the Euclidean distance between the mapped received translated vector and the candidate symbols of the first quadrant (positive sector).

(iii) Now that the optimum non-negative symbol (i.e. one that is in the first quadrant (positive sector)) is found, the proper sign variation of this is determined by finding the sgn function of each co-ordinate of the translated received vector, and then multiplying (element-by-element) the optimum non-negative symbol by this. These steps will produce the optimum symbol to be used in Step 4.

This detection scheme was first applied in [3] to two-dimensional hexagonal constellations and the four-dimensional constellations of Gersho and Lawrence [1].

Application of the detection method to 8D-L

The detection scheme above can be applied to 8D-L, since this constellation can be thought of as being made up of stacking hypercubes. The vertices of the hypercube at the origin are given by [1 1 1 1 1 1 1 1], and all of its sign variations. This hypercube encloses a point at the origin (which is not really part of the constellation) and the fourteen permutations of [1 1 1 1 0 0 0 0], and their sign variations. Therefore, the candidate symbols in Step 2 above have 240 members. Fortunately, this number can be drastically reduced to only sixteen by the procedure given by Step 5. It should be clear that the candidate non-negative symbols are given by the origin, the fourteen permutations of [1 1 1 1 0 0 0 0], and [1 1 1 1 1 1 1 1]. Thus, the minimum of only sixteen distances has to be computed each block period.

The centers of the hypercubes that are adjacent to the one at the origin are given by [0 0 0 0 0 0 0 2], [0 0 0 0 0 0 2 2], [0 0 0 0 0 2 2 2], [0 0 0 0 2 2 2 2], [0 0 0 2 2 2 2 2], [0 0 2 2 2 2 2 2], [0 2 2 2 2 2 2 2], [2 2 2 2 2 2 2 2], and all permutations and sign variations of these. (Note that the centers of adjacent hypercubes are at a distance of two apart). Fortunately, all the constellation points for 8D-L, L $\leq$ 16, are generated by the hypercubes with these centers, and so the quantizers of Step 1 need only have three levels in each co-ordinate. Thus, if a co-ordinate of the received vector is less than -1, then the quantizer for that co-ordinate will have a value of -2 on its output. If it is between -1 and 1, then the output will be 0, and if it is greater than 1, the quantizer's output will be 2. Thus, the centers of the hypercubes are easily established.

Note that for L > 16, the level of quantization will be higher since there will be centers at other locations, e.g. at [0 0 0 0 0 0 0 4].

Detecting 8D-8B

To detect 8D-8B optimally, it is not necessary to do Steps 1, 2, and 4 in the above detection process. Only Step 3 as simplified in Step 5 is needed, with the following modifications:

(i) replace the "received translated vector" by just the "received vector",

(ii) replace the "mapped received translated vector" by the "mapped received vector", and

(iii) the "candidate symbols of the positive sector" are now given by the twenty-six entries of Table 2 for 8D-8B. Thus, the minimum of the distances between these 26 points and the mapped received vector must be found every block period.

Performance Evaluation

In this section, the probability of symbol error P_{es} for 8D-L will be found and compared to that of regular M-QAM over four symbol periods. One measure which can be used for this purpose is the Asymptotic Estimate [4,5], which states:

$$P_{es} = \frac{K}{2} \operatorname{erfc}\left(d_{min}/\sqrt{8}\sigma\right) \qquad (2)$$

where d_{min} is the minimum Euclidean distance of the constellation,

σ^2 is the noise variance in each co-ordinate, and is related to the SNR in the following way:

$$\sigma^2 = P/(8*SNR) \qquad (3)$$

K is the error coefficient, which can be found from

$$K = \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{M} \delta(\|p_i - p_j\| - d_{min}) \qquad (4)$$

where $\|p_i - p_j\|$ is the Euclidean distance between symbols p_i and p_j,

$$\delta(x) = 1 \text{ if } x = 0, \quad = 0 \text{ otherwise.}$$

For 8D-L, d_{min} is 2 by design. Also, K is easily upper bounded to be 240 for all L since $\|p_i - p_j\| = 2$ only when $p_i - p_j$ is given by the fourteen permutations of [1 1 1 1 0 0 0 0], and the eight permutations of [0 0 0 0 0 0 0 2], and their sign variations.

Using (4), it is found that K = 58.5 for 8D-8A, K = 53.47 for 8D-8B, and K = 117.1 for 8D-12. Unfortunately, K for 8D-16 was not evaluated since the computational task was too much for the author's bottom of the line, ancient personal computing system. Note that these error coefficients are attainable by 8D-L only if they are detected optimally, as is the case of 8D-8B. It has already been stated that the detection method above causes these coefficients to be increased (because it assumes that no symbols are missing from the stacking hypercubes). Thus, these optimum coefficients can be viewed as lower bounds for the true error coefficients. Fortunately, 240 still remains to be a valid upper bound, and so reasonable estimates for the P_{es} performance for 8D-L can be found analytically.

Figure 2 shows the P_{es} performance for 8D-8A, 8D-8B, and QPSK over four symbol periods. As can be seen, 8D-8B achieves the best performance, having a gain of 0.37 dB over 8D-8A, and 1.78 dB over QPSK. Likewise, Fig. 3 shows the P_{es} performance for 8D-12, 8D-16, and 16-QAM. Clearly, 8D-12 provides a 3.17 dB gain over 8-QAM. One reason for this large gain is that 8-QAM is not an efficient (as far as SNR is concerned) two-dimensional constellation. So, when it is compared to an efficient eight-dimensional constellation, the improvement is dramatic.

It can also be seen from Fig. 3 that 8D-16 achieves a 2.55 dB gain over 16-QAM. This is slightly better than that obtained with the eight-dimensional constellation of Gersho and Lawrence. (The gains mentioned in this and the previous paragraphs are calculated with (2) for $P_{es} = 10^{-6}$).

Figures 2 and 3 were prepared with the help of the information in Table 5, but it should be pointed out that K = 240 was used in preparing the curves for 8D-8A, 8D-12, and 8D-16. This was done because no analytical method was presented to find the error co-efficients for the 8D-L detector. Also, it will be seen that K = 240 is indeed a very tight bound. Furthermore, the theoretical gain of 8D-L over the corresponding M-QAM system is also shown in Table 5. This gain is simply $10 \log (P_1/P_2) - .2 \log_2 (K_1/K_2)$, where P_1 is the average energy of 8D-L and P_2 is the average energy of the corresponding M-QAM system. K_1 is the error co-efficient of 8D-L; for 8D-8B, it is 53.47, and 240 otherwise. K_2 is the error co-efficient of the corresponding M-QAM system. As can be seen, the theoretical gains show reasonable to excellent agreement with the values obtained from Figs. 1 and 2.

To see how well (2) estimates P_{es} performance, Monte Carlo simulations, using the Importance Sampling technique [6,7] were performed, and the results are also shown in Figs. 2 and 3. As can be seen, the simulation points for 8D-8B are very near the predicted curve. Therefore, this verifies that the detection scheme for 8D-8B is valid since it achieves the optimum error co-efficient.

On the other hand, the simulation points for 8D-8A, 8D-12, and 8D-16 are close to the predicted curves which were drawn assuming K = 240. Therefore, this shows that K = 240 is a very tight upper bound. Furthermore, this indicates that the detection scheme does not achieve optimum detection, but rather increases the error co-efficients from the optimum values given in Table 5. This behavior was expected as noted earlier. This loss due to non-optimum detection can be quantified by use of Forney's rule [2] given earlier. Thus, the loss is given by $0.2 \log_2(240/K)$: for 8D-8A, it is 0.407 dB, and for 8D-12, it is 0.207 dB. For 8D-16, the loss is expected to be no greater than it is for 8D-12. These losses are not to be confused with the error co-efficient loss column in Table 5, which is the loss that occurs because the error co-efficient of 8D-L is greater than that of the corresponding M-QAM system.

Conclusion

In this paper, eight-dimensional constellations have been proposed that appreciably outperform the corresponding M-QAM systems. Simple detectors were also proposed for these constellations. Monte Carlo simulations were done which confirmed the theoretical predictions of the P_{es} performance of these new constellations. Future work should concentrate on finding methods of reasonable complexity of mapping the incoming L data bits into the 2^L symbols of these constellations at the transmitter, and performing the corresponding inverse mapping at the receiver.

References

1] Gersho, A., and V. Lawrence, "Multi-dimensional signal constellations for voiceband data transmission," *IEEE J. Selected Areas in Commun.*, vol. SAC-2, pp. 687-702, Sept. 1984.

2] Forney, G. D., Jr., "Coset codes I: Introduction and geometrical classification," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1123-1151, Sept. 1988.

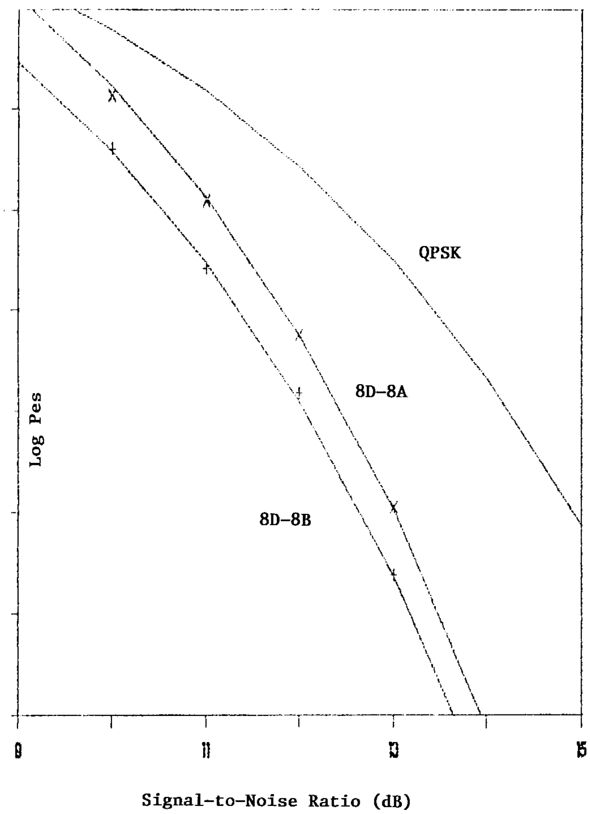
3] Cartwright, K. V., "Digital transmission in the presence of depolarization crosstalk with four-dimensional constellations," Ph.D dissertation, Department of Electrical Engineering, Tulane University, New Orleans, La., May 1990.

4] Benedetto, S., M. Marsan, G. Albertengo, and E. Giochin, "Combined coding and modulation: Theory and applications," *IEEE Trans. Info. Theory*, vol. 34, pp. 223-236, March 1988.

5] Welti, G. R., "PCM/FDMA satellite telephony with 4-dimensionally-coded Quadrature Amplitude Modulation," *COMSAT Tech. Rev.*, vol. 6, pp. 323-338, Fall 1976.

6] Shanmugan, K., and P. Balaban, "A modified Monte Carlo simulation technique for the evaluation of error rate in digital communication systems," *IEEE Trans. Commun.*, vol. COM-28, pp. 1916-1924, Nov. 1980.

7] Orsak, G., and B. Aazhang, "On the theory of Importance Sampling applied to the analysis of detection systems," *IEEE Trans. Commun.*, vol. COM-37, pp. 332-339, April 1989.



g.2. Probability of symbol error for 8D-8A, 8D-8B, and 'SK over four symbols.

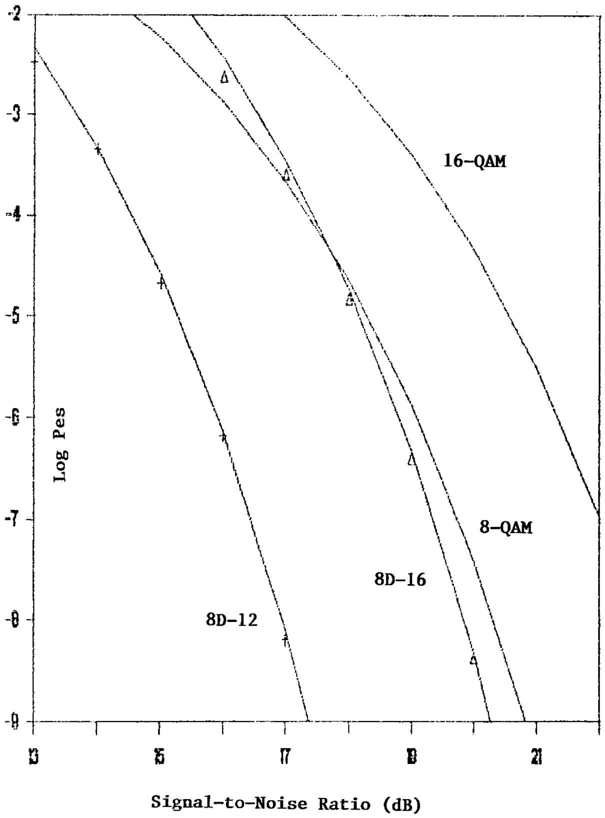


Fig.3. Probability of symbol error for 8D-12, 8D-16, 8-QAM and 16-QAM over four symbols.

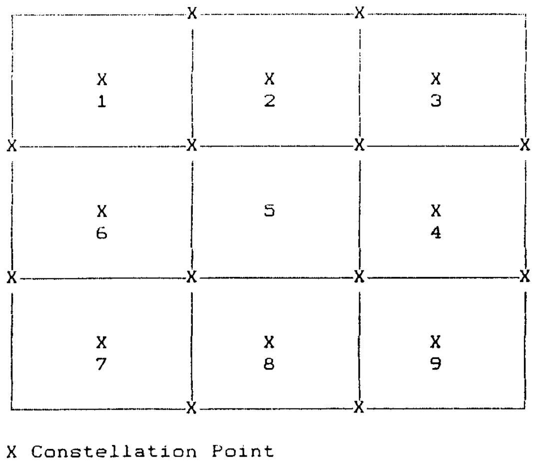


Fig.1. Fictitious two-dimensional constellation used to describe the detection process for 8D-L. The numbers represent the number of the stacking rectangle.