Fourth Power Phase Estimation with Alternative Two-Dimensional Odd-Bit Constellations

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***Abstract*—It is well established that the fourth power phase estimator does not perform well for QAM cross (i.e., odd-bit) constellations. In this letter, new two-dimensional rectangular constellations are presented that give much improved performance, at a reasonably low increase in required constellation energy. For example, the 128B-COB constellation reduces the variance by an order of magnitude, while increasing the constellation's average energy by only 0.28 dB and the peak to average energy ratio by 1.25 dB, relative to 128-QAM.**

***Index Terms*—Carrier phase estimation, constellation design, modulation, quadrature amplitude modulation, synchronization.**

I. Introduction

The NONDATA aided fourth power phase estimator was shown by Moeneclaey and de Jonge [1] to be optimal in the sense of maximum likelihood in the limit of small signal-to-noise ratios (SNR's) for arbitrary two-dimensional rotationally symmetric constellations, such as square and cross QAM constellations. An approximate expression for the variance of the estimation was also derived in [1], and is given in (3). Using this expression, Georghiades [2] demonstrated that the variance of the fourth power phase estimator is extremely large for cross constellations, and unfortunately, it does not improve significantly as SNR increases due to the self-noise of the constellation, which is measured by the parameter B in (3).

Georghiades [2] recognized that perhaps the performance of the power-law estimator might be improved by constraining the transmitted symbols in some way. He suggested that this might involve loss of throughput and possible increase in complexity. He therefore, decided to modify the received constellation instead, but concluded that for cross constellations this technique provided no substantial relief of the large variance problem.

It is the purpose of this letter to point out that two-dimensional rectangular constellations can be designed for the transmission of an odd number of bits that produce variances on the same order of magnitude as that produced by square QAM constellations. The price paid for this substantial reduction in variance is a small increase in energy needed by the constellation. In fact, constellations can be designed that trade-off amount of energy increase with respect to variance generated by the fourth power phase estimator.

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II. Review of the Power-Law Phase Estimator

The following review borrows heavily from [2], where it is assumed that the system is already equalized and frequency-synchronized, and that timing and relative gain control have been achieved. Given these assumptions, the baud-rate samples of the output of a matched filter are given by

$$r_k = d_k e^{j\phi} + n_k, \qquad k = 1, 2, \cdots, L, \cdots \tag{1}$$

where d_k is a complex number that represents the QAM symbol transmitted at time kT; $1/T$ is the signaling rate; ϕ is the unknown phase offset that is to be estimated; and n_k are complex independent identically distributed (i.i.d) zero-mean Gaussian random variables with independent real and imaginary parts having variance σ^2. Furthermore, the average constellation energy is given by $W = E[|d_k|^2]$. In this section, it is assumed that W is normalized to unity. Hence, the symbol signal-to-noise ratio (SNR) is given by $1/2\sigma^2$.

The Pth power phase estimator produces a phase estimate from the following:

$$\hat{\phi} = \frac{1}{P} \arg\left[E\left[d^{*P}\right] \sum_{k=1}^{L} r_k^P\right]. \tag{2}$$

Moeneclaey and de Jonge [1] also provided an approximate expression for the variance of the estimation error, which is given below:

$$\operatorname{var}\left(\hat{\phi} - \phi\right) = \left(\frac{A}{2\text{SNR}} + B\right)\frac{1}{L} \tag{3}$$

where L is the length of the observed data block, and

$$A = \frac{E\left[|d^2|\right] E\left[|d^2|^{P-1}\right]}{\left|E\left[d^P\right]\right|^2} \geq 1 \tag{4}$$

$$B = \frac{2\left|E\left[d^P\right]\right|^2 E\left[\left|d^2\right|^P\right] - E^2\left[d^P\right] E\left[d^{*2P}\right] - E^2\left[d^{*P}\right] E\left[d^{2P}\right]}{4P^2 \left|E\left[d^P\right]\right|^4}. \tag{5}$$

For QAM, $P = 4$. Also, as was pointed out in [1], $B = 0$ for PSK constellations, but is nonzero for QAM constellations, thereby causing a nonzero phase-error variance even at infinite SNR. In fact, as demonstrated in [2], A and B are excessively large for cross QAM constellations.

III. M-COB Constellations

In this section, alternative two-dimensional odd-bit constellations, called M-COB constellations, will be described. Although

TABLE I
ADDITIVE AND SELF-NOISE COEFFICIENTS, AND ENERGY REQUIREMENTS FOR VARIOUS ODD-BIT CONSTELLATIONS

Constellation	A	B	Average Energy Wc	Increase* in Average Energy (dB)	Peak Energy Wp	Peak-to-Average Energy Ratio R	Increase* In R (dB)
32-QAM	52.6316	3.1413	20.00	NA	34.00	1.700	NA
32-COB	5.9643	0.1498	24.00	0.7918	50.00	2.083	0.8824
128-QAM	62.0675	3.7929	82.00	NA	170.0	2.073	NA
128A-COB	3.1139	0.0980	91.50	0.4761	242.0	2.645	1.058
128B-COB	6.2996	0.1243	87.50	0.2819	242.0	2.766	1.252
128C-COB	15.9354	0.5064	84.00	0.1047	242.0	2.881	1.429
512-QAM	64.6815	3.9818	330.0	NA	754.0	2.285	NA
512A-COB	3.9128	0.0969	354.5	0.3110	1058	2.984	1.159
512B-COB	6.3494	0.1491	344.5	0.1868	1058	3.071	1.284
512C-COB	9.4916	0.2644	338.0	0.1040	1058	3.130	1.367
2048-QAM	65.352	4.0308	1322.0	NA	3362	2.543	NA
2048A-COB	5.1483	0.1748	1383.0	0.1959	3874	2.801	0.4197
2048B-COB	6.2962	0.1869	1369.8	0.1543	3874	2.828	0.4613
2048C-COB	9.1608	0.3454	1344.5	0.0733	3874	2.881	0.5420

*Relative to the correspondencing M-QAM constellation.

these constellations were arrived at heuristically, they have A and B values substantially smaller than those of cross QAM constellations. The price paid for this improved performance is a small increase in both the average energy and the peak to average energy ratio required by the M-COB constellation. The peak to average energy ratio is an important parameter as it determines the efficiency of the power amplifier [3].

Let a constellation point, i.e. a symbol, of the M-COB constellation, be described by $c_i = \{a_i, b_i\}$. Then, by design, the minimum distance to any other constellation point is 2. With this choice, the components of c_i take on odd or even integer values. Hence, the average energy of the constellation will not be unity, but rather it is given by W_c. The normalized transmitted symbol will then be given by $d_k = (a_k + jb_k)/\sqrt{W_c}$.

If the constellations below are plotted, one can see that the constellations have been made as square as possible.

A. The 32-COB Constellation

This constellation consists of the following points: $\{1,1\}, \{1,3\}, \{1,5\}, \{3,5\}, \{5,5\}$ and all permutations and sign variations of both these points and their permutations. For example, $\{3,5\}$ has $\{5,3\}$ as its permutation and sign variations of $\{\pm 3, \pm 5\}$ and its permutation has sign variations of $\{\pm 5, \pm 3\}$. Therefore, $\{3,5\}$ gives rise to 8 symbols when its permutation and all sign variations are accounted for.

B. The 128-COB Constellations

Here, three different constellations for $M = 128$ will be given to show the tradeoff between low values of A and B and the average constellation energy W_c, the peak energy W_p, and the peak to average energy ratio R. All three constellations will have 100 symbols described by $\{a, b\}$ where a, b are odd integers between -9 and 9, inclusive. The remaining 28 symbols will be distributed equally in each quadrant.

1) The 128A-COB Constellation: For this constellation, the 7 symbols in the first quadrant are given by $\{11,5\}$, $\{11,7\}, \{11,9\}, \{5,11\}, \{7,11\}, \{9,11\}$ and $\{11,11\}$. The remaining 21 symbols are just sign variations of these.

2) The 128B-COB Constellation: For the 128B-COB constellation, the 7 symbols in the first quadrant are given by $\{11,1\}, \{11,3\}, \{11,9\}, \{1,11\}, \{3,11\}, \{9,11\}$, and $\{11,11\}$. As before, the remaining 21 symbols are just sign variations of these.

3) The 128C-COB Constellation: Finally, for the 128C-COB constellation, the 7 symbols in the first quadrant are given by $\{11,1\}, \{11,3\}, \{11,5\}, \{1,11\}, \{3,11\}, \{5,11\}$, and $\{11, 11\}$. As before, the remaining 21 symbols are just sign variations of these.

C. The 512-COB Constellations

As was the case for $M = 128$, three different constellations for $M = 512$ will be given. All three constellations will have 484 symbols described by $\{a, b\}$ where a, b are odd integers between -21 and 21, inclusive. The remaining 28 symbols will be distributed equally in each quadrant.

1) The 512A-COB Constellation: For this constellation, the seven symbols in the first quadrant are given by $\{23, 17\}, \{23, 19\}, \{23, 21\}, \{17, 23\}, \{19, 23\}, \{21, 23\}$, and $\{23, 23\}$. The remaining 21 symbols are just sign variations of these.

2) The 512B-COB Constellation: For the 512B-COB constellation, the seven symbols in the first quadrant are given by $\{23, 1\}, \{23, 3\}, 23, 21, \{1, 23\}, \{3, 23\}, \{21, 23\}$, and $\{23, 23\}$. As before, the remaining 21 symbols are just sign variations of these.

3) The 512C-COB Constellation: Finally, for the 512C-COB constellation, the seven symbols in the first quadrant are given by $\{23, 1\}, \{23, 3\}, \{23, 5\}, \{1, 23\}, \{3, 23\}, \{5, 23\}$, and $\{23, 23\}$. As before, the remaining 21 symbols are just sign variations of these.

D. The 2048-COB Constellations

This constellation has 1936 symbols which are described by $\{a, b\}$ where a, b are odd integers between -43 and 43, inclusive. The remaining 112 symbols are equally distributed to each quadrant. So, the remaining symbols are given by $\{45, a\}$ and $\{a, 45\}$, and sign variations of these, where the values a is allowed defines the 2048A, B, and C variations.

1) The 2048A-COB Constellation: For this constellation, $15 < a < 45$ and a is an odd integer.

2) The 2048B-COB Constellation: For this constellation, a is a positive odd integer less than 15 or satisfying $29 < a < 45$.

3) The 2048C-COB Constellation: For the 2048C-COB constellation, a is a positive odd integer less than 29.

IV. Performance of the M-Cob Constellations

To find the performance as far as the fourth power phase estimator is concerned, values of A and B for each constellation were determined with (4) and (5) (with $P = 4$, of course). These are shown in Table I, along with the constellation energy requirements.

As can be seen, the A and B values for the M-COB constellations are significantly less than the corresponding M-QAM constellations, thereby testifying to the greatly improved performance of the fourth power phase estimator for the M-COB constellations. As is also evident, there is a price to pay for this improved performance—an increase in the average energy and an increase in the peak to average energy needed by the constellation. However, as Table I shows, this price is reasonable, especially for the MB-COB constellations, which provide a good compromise between performance and increase in average and peak-to-average energy needs.

V. Conclusion

In this letter, alternative two-dimensional odd-bit constellations, called M-COB constellations, have been described which give greatly improved performance over cross QAM constellations for the fourth power phase estimator. For each M, $M > 32$, three different versions have been given which illustrates the tradeoff possible between performance and energy costs.

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