

AN EIGHT-DIMENSIONAL PHASE INVARIANT TRELLIS CODED
MODULATION SCHEME WITH CONSTANT ENVELOPE¹

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ABSTRACT

This paper presents an eight-dimensional (8D) Trellis Coded Modulation (TCM) scheme which uses 512 points to transmit 2 information bits per signaling interval. The system has constant envelope and with only eight states on the trellis, yields an asymptotic gain of 6.02 dB without expanding the bandwidth over that of the equivalent uncoded system (QPSK over four consecutive time intervals). The size of the coded constellation is doubled but the Minimum Squared Euclidean Distance (MSED) and the peak and average energies remain identical to those of the uncoded system, therefore avoiding the 3 dB loss in Ungerboeck's conventional codes. The novel constellation is formed by the union of an 8D-QPSK and a rotated version of it. Our new scheme achieves an effective gain of 5.64 dB, a small normalized redundancy of $\rho=0.25$, has constant envelope, achieves spectral efficiency of 2b/s/Hz if Nyquist filtering is used, and uses simple and small constituent 2D constellations. The scheme is also invariant under phase rotations of 90°, 180° and 270°. Extensive comparisons of our code to other conventional and non-conventional codes are given.

I. INTRODUCTION

Since the publication of Ungerboeck's paper [1] on Trellis Coded Modulation (TCM) much work has been done trying to find more and better ways to achieve improved system probability of error performance by combining modulation and coding without sacrificing data rate or increasing the bandwidth of the system. In this paper we present one such scheme which efficiently uses 512 points with unit energy per dimension in an eight-dimensional (8D) space.

Many applications which use nonlinear amplifiers and repeaters call for the use of constant envelope modulation schemes. Some systems, such as CPM [2], have attracted the attention of many researchers, while others, like Q²PSK [3] have recently been proposed. Older systems, which include M-PSK and FSK and are not as spectrally compact, are currently in use and have been studied extensively. The constant envelope system we study in this paper achieves high gains with spectral efficiency of 2b/s/Hz if Nyquist filtering is used.

With Ungerboeck's 2D TCM schemes [1] q information bits are sent in each signaling interval using a constellation of 2^{q+1} points. Wei [4] proposed using ND constellations ($N>2$) with rate $m/(m+1)$ convolutional encoders to reduce the 3 dB cost of signal expansion in Ungerboeck codes. Wei's multidimensional scheme transmits q information bits per signaling interval using a 2L-dimensional constellation of 2^{Lq+1} points.

When multidimensional constellations are used fewer redundant bits are added for each 2D signaling interval and the normalized redundancy can thus be reduced below one. The normalized redundancy of the code presented here is $\rho=0.25$. Both Wei [4] and Kaminsky [5] consider the reduction in the size of the constituent lower dimensional constellations an advantage of multidimensional TCM.

The performance improvement of our scheme over conventional TCM systems is obtained by increasing the Minimum Square Euclidean Distance (MSED) between coded sequences without increasing the modulation level, therefore maintaining the average energy identical to that of the uncoded base system (QPSK over four time slots). A similar idea was used in [5], where the number of channels was increased by the use of dual polarization.

Ungerboeck proposed in [1] the method of mapping by set partitioning to assign signal points from an expanded 2D constellation to the branches of the trellis of a convolutional encoder. Wei [4] proposed a different partitioning method for multidimensional constellations, based on partitions of the constituent 2D constellations. We follow Wei's partitioning method to obtain the necessary subsets of points and then apply Ungerboeck's and Wei's rules to assign the subsets to the branches of the trellis. Other partitioning methods are given in [6] and [7], based on lattices and cosets and block coding, respectively.

Several authors [4,6,7,8] have presented 8D TCM schemes that have considerable asymptotic gains but, to the best of our knowledge, none of them achieve 6 dB with only eight states. Also, most 8D schemes were used to transmit information at higher rates. An MLSE (Viterbi decoder) must be used at the receiver for optimum detection of our code, but since the trellis has only eight states the complexity of the decoder is quite reasonable.

The rest of the paper is organized as follows: In section II the low energy 8D constellation X8 and its characteristics are presented. In section III we follow Wei's partitioning method to obtain 32 subsets of points (sublattices) with large intrasubset MSED. In section IV we describe the TCM system which uses a convolutional encoder of rate $R_c=2/3$ to transmit, in four consecutive time intervals ($N_T=4$), information at a rate of 2 bits/T. In section V the phase invariance of X8-TCM is discussed. In section VI results are presented and the system compared to others previously published. Conclusions and suggestions for further work are given in section VII, followed in section VIII by selected references.

II. THE EIGHT DIMENSIONAL CONSTELLATION X8

Figure 1 describes the system we propose to use. We form an 8D transmitted symbol by operating on eight input bits at a time. The mapper changes the input bit stream into the in-phase (I) and quadrature-phase (Q) symbol streams. Typical waveforms are illustrated in Fig. 2.

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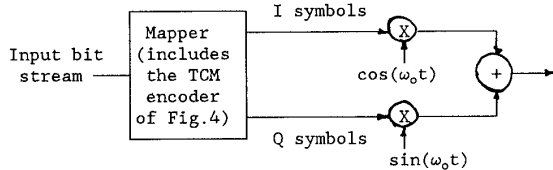


Fig. 1: Block diagram of the X8 modulator. The I and Q streams taken over four time slots form the 8D transmitted symbol.

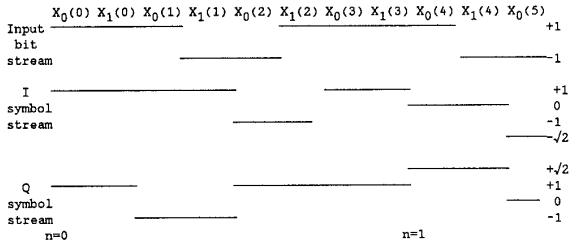


Fig. 2: Typical bit and symbol streams for the modulator of Fig. 1.

We operate on four consecutive symbol time slots for each of the I and Q streams to form the transmitted symbol. If $X_0(4n+j)$ and $X_1(4n+j)$, $j=0,1,2,3$, represent the value of the I and Q data streams for the j th time slot, respectively, the i th transmitted symbol can be written as $p_i = (p_{20}, p_{21}, p_{22}, p_{23})$, where p_{2j} is the 2D symbol given by $(X_0(4n+j), X_1(4n+j))$ $j=0,1,2,3$. For regular QPSK over four time slots, there are 256 unique values of p_i . This follows since p_{2j} can be $(1,1)$ or any sign variation of it. However, we need 512 symbols for our TCM scheme, and therefore need an additional 256 symbols to allow for code redundancy. Furthermore, to meet the requirement of constant envelope for all time, we require that $X_0^2(4n+j) + X_1^2(4n+j) = 2$, $j=0,1,2,3$. Fortunately, we can fulfill both these requirements without reducing the MED of the 8D constellation or increasing the peak or average energy of the constellation. To do this we simply allow 8D symbols of the form $p_{iR} = (p_{20R}, p_{21R}, p_{22R}, p_{23R})$, where p_{2jR} is p_{2j} above rotated by 45 degrees ($j=0,1,2,3$). Hence, p_{2jR} can take on the values $(0, \sqrt{2})$, $(\sqrt{2}, 0)$, $(0, -\sqrt{2})$ and $(-\sqrt{2}, 0)$. Since p_{2jR} and j take on four values each, p_{iR} can be one of 4⁴ unique symbols.

The constellation X8 is the union of (p_i) and (p_{iR}) . The MED between any two symbols of the rotated 8D constellation (p_{iR}) is two, the same as that of (p_i) . Furthermore, the MED between any symbol p_i and any symbol p_{kR} is $4+4(1-\sqrt{2})^2$. Therefore, the MED of X8 is 2, the same as that of regular QPSK over four time slots. Notice also that the peak and average energy of X8 is 8. Hence, we can transmit 512 symbols with the same energy that regular QPSK over four consecutive time slots needs for only 256 symbols.

Although Wei [4] reduces the normalized constellation expansion factor by using multidimensional constellations, he still increases the modulation level by use of the added outer points. The additional points in the constellation we use are at the same level as the points used in the uncoded system, thus avoiding any loss due to average energy increase. This is what makes our system superior to others.

III. PARTITION OF THE X8 CONSTELLATION

Several partitioning methods are available to us; we chose to follow Wei's method [4] due mainly to its simplicity. It is intuitive too since the constellation

we use is obviously constituted by 2D QPSK constellations.

We start by partitioning each constituent QPSK constellation (2D lattice³) into four 2D sublattices (A, B, C and D) of one point each, as shown in Fig. 3:

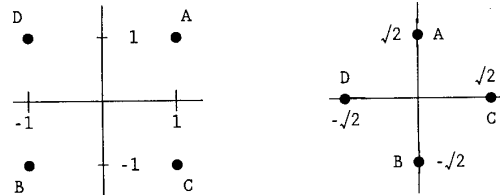


Fig. 3: Set partitioning of the constituent QPSK constellations.

If the QPSK constellations are thought of as being part of the lattice Z^2 [6] the MSD between the sublattices is easily shown to be 16. Now we form sixteen 4D types for each of the two constituent 2D sublattices by concatenating a pair of 2D points:

- (A,A), (A,B), (A,C), (A,D)
- (B,A), (B,B), (B,C), (B,D)
- (C,A), (C,B), (C,C), (C,D)
- (D,A), (D,B), (D,C), (D,D)

From these sixteen types we construct eight 4D sublattices (for each of the two groups) by grouping together two types such that the MSD of the 4D sublattices is still 16:

- 0: (A,A), (B,B) 4: (A,C), (B,D)
- 1: (C,C), (D,D) 5: (C,B), (D,A)
- 2: (A,B), (B,A) 6: (A,D), (B,C)
- 3: (C,D), (D,C) 7: (C,A), (D,B)

We proceed to form two groups of sixty-four 8D types each, by concatenating elements from the above 4D types:

- (0,0), (0,1), ..., (0,7)
- (1,0), (1,1), ..., (1,7)
- ...
- (7,0), (7,1), ..., (7,7)

We further group the 64 8D types from each group into 16 8D sublattices with MSD 16. We denote these by S_0 through S_{15} for the group with coordinates ± 1 , and S_{16} through S_{31} for the group with coordinates $(0, \pm \sqrt{2})$ or $(\pm \sqrt{2}, 0)$:

- S_0 or S_{16} : (0,0), (1,1), (2,2), (3,3)
- S_1 or S_{17} : (0,1), (1,0), (2,3), (3,2)
- S_2 or S_{18} : (0,2), (1,3), (2,0), (3,1)
- S_3 or S_{19} : (0,3), (1,2), (2,1), (3,0)
- S_4 or S_{20} : (4,4), (5,5), (6,6), (7,7)
- S_5 or S_{21} : (4,5), (5,4), (6,7), (7,6)
- S_6 or S_{22} : (4,6), (5,7), (6,4), (7,5)
- S_7 or S_{23} : (4,7), (5,6), (6,5), (7,4)
- S_8 or S_{24} : (0,4), (1,5), (2,6), (3,7)
- S_9 or S_{25} : (0,5), (1,4), (2,7), (3,6)
- S_{10} or S_{26} : (0,6), (1,7), (2,4), (3,5)
- S_{11} or S_{27} : (0,7), (1,6), (2,5), (3,4)
- S_{12} or S_{28} : (4,0), (5,1), (6,2), (7,3)
- S_{13} or S_{29} : (4,1), (5,0), (6,3), (7,2)
- S_{14} or S_{30} : (4,2), (5,3), (6,0), (7,1)
- S_{15} or S_{31} : (4,3), (5,2), (6,1), (7,0)

³ The constellation formed by $(\pm \sqrt{2}, 0)$ and $(0, \pm \sqrt{2})$ is not strictly a lattice, but we will refer to it as one to maintain Wei's notation.

For example, the sublattice S_{16} contains the following 16 points with MSED 16 between them:

- (0,0): (A,A,A,A) = (0,√2,0,√2,0,√2,0,√2)
 (A,A,B,B) = (0,√2,0,√2,0,-√2,0,-√2)
 (B,B,A,A) = (0,-√2,0,-√2,0,√2,0,√2)
 (B,B,B,B) = (0,-√2,0,-√2,0,-√2,0,-√2)
- (1,1): (C,C,C,C) = (√2,0,√2,0,√2,0,√2,0)
 (C,C,D,D) = (√2,0,√2,0,-√2,0,-√2,0)
 (D,D,C,C) = (-√2,0,-√2,0,√2,0,√2,0)
 (D,D,D,D) = (-√2,0,-√2,0,-√2,0,-√2,0)
- (2,2): (A,B,A,B) = (0,√2,0,-√2,0,√2,0,-√2)
 (A,B,B,A) = (0,√2,0,-√2,0,-√2,0,√2)
 (B,A,A,B) = (0,-√2,0,√2,0,√2,0,-√2)
 (B,A,B,A) = (0,-√2,0,√2,0,-√2,0,√2)
- (3,3): (C,D,C,D) = (√2,0,-√2,0,√2,0,-√2,0)
 (C,D,D,C) = (√2,0,-√2,0,-√2,0,√2,0)
 (D,C,C,D) = (-√2,0,√2,0,√2,0,-√2,0)
 (D,C,D,C) = (-√2,0,√2,0,-√2,0,√2,0)

Finally, we form two groups, F and G, of eight families of four 8D sublattices each, reducing the MSED within each family to eight. Any family F_i can have only one member in common with a family G_j , $i,j=0,1,\dots,7$:

- $F_0 = 0, 1, 2, 3$ $F_4 = 16, 17, 18, 19$
 $F_1 = 8, 9, 10, 11$ $F_5 = 24, 25, 26, 27$
 $F_2 = 4, 5, 6, 7$ $F_6 = 20, 21, 22, 23$
 $F_3 = 12, 13, 14, 15$ $F_7 = 28, 29, 30, 31$

and

- $G_0 = 0, 4, 18, 22$ $G_4 = 8, 14, 26, 28$
 $G_1 = 1, 5, 19, 23$ $G_5 = 9, 15, 27, 29$
 $G_2 = 2, 6, 16, 20$ $G_6 = 10, 12, 24, 30$
 $G_3 = 3, 7, 17, 21$ $G_7 = 11, 13, 25, 31$

As we will see in the next section the families F and G will be assigned to transitions diverging from, or merging into a trellis state, respectively.

IV. THE X8-TCM SCHEME

For transmission with spectral efficiency of 2 information bits per signaling interval (equivalent to 2 b/s/Hz if Nyquist filtering is used), our TCM system consists of a convolutional encoder of rate $R_c=2/3$ and a QPSK modulator over four consecutive time intervals.

Two of the eight information bits arriving every four signaling intervals (I_6 and I_7) enter the convolutional encoder and three coded bits are produced at its output (O_6 , O_7 and O_8). These three coded bits together with two uncoded bits (I_4 and I_5) select one of the 32 8D sublattices while the remaining four uncoded information bits (I_0 through I_3) determine which of the 16 points from within the selected subset will be used for transmission. The system is shown in Fig. 4.

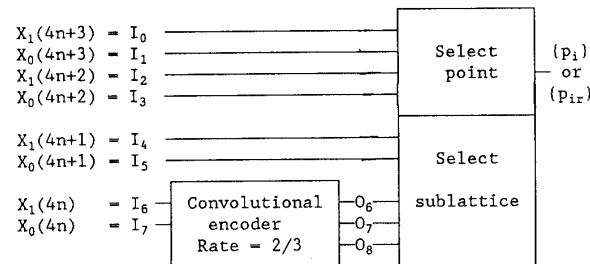


Fig.4: Block diagram of the X8-TCM scheme; n is the time index and p the selected point.

The convolutional encoder chosen, shown in Fig. 5, is of the feedback type, has rate $R_c=2/3$ and constraint length $\nu=3$. The trellis diagram is shown in Fig. 6, where the states are named according to the contents of the memory cells and the signals associated with the transitions diverging from or merging into each state are shown beside the respective states in a top down fashion.

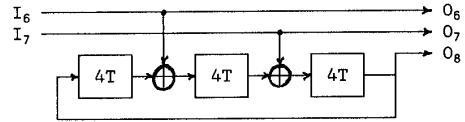


Fig. 5: Feedback convolutional encoder of $R_c=2/3$ and $\nu=3$ for the X8-TCM scheme.

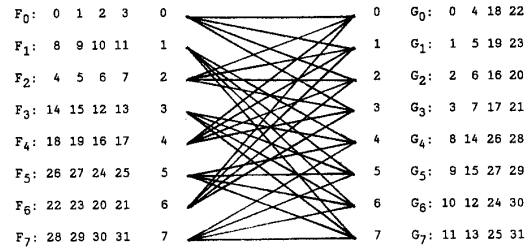


Fig. 6: Trellis diagram of the X8-TCM scheme. Each line represents 16 parallel branches.

The idea behind the assignment of points to branches of the trellis is to maximize the free distance of the code. In our case this distance is 16, the MSED between points in any of the 8D sublattices. Minimization of the error coefficient was also considered. The assignment satisfies the following Ungerboeck-Wei rules of mapping [1,4]:

- 1) Points with maximum MSED are assigned to parallel transitions, 2) The 8D sublattices associated with transitions diverging from, or merging into a trellis state, are different from each other and have the following maximum possible ED, and 3) The 8D sublattices associated with transitions diverging from, or merging into a trellis state, belong to the same 8D family F_i or G_i , $i=0,1,\dots,7$, respectively.

Satisfaction of rules 2 and 3 poses no difficulty since we have as many sublattices as non-parallel transitions in the trellis, and as many families as nodes.

V. PHASE INVARIANCY

The phase transparency of the code becomes important when the system is susceptible to phase rotations. Differential encoding is generally used to make the codes invariant to phase rotations [4].

Each sublattice S_0-S_{31} used in the X8-TCM scheme is invariant under 90° rotations, i.e. if each 2D constituent point in lattice S_i is rotated by 90° , another 8D point in the same lattice S_i is obtained, $i=0,1,\dots,31$. This means that X8-TCM is invariant under rotations of integer multiples of 90° . For example, if the 8D point (1,1,1,1,1,1,1,1) which belongs to sublattice S_0 is rotated by 90° , we obtain the point (-1,1,-1,1,-1,1,-1,1) which also belongs to S_0 .

VI. RESULTS

The MSED between any pair of paths for the system presented in the last section is 16 due to the parallel

transitions as well as some paths of length 2. The free squared distance of the code is:

$$d_{free}^2 = \min\{d_1^2, d_{min}^2\} = \min\{16, 16\} = 16 \quad (1)$$

The MSSED of the uncoded 8D QPSK system needed to transmit 2 b/s/Hz is $d_u^2 = 4$. The average (and peak) energy of both coded and uncoded systems is $E = 8$. The asymptotic gain of the X8 system is therefore

$$G_a = 10 \log_{10} \left(\frac{d_{free}^2}{d_u^2} \right) = 10 \log_{10} \left(\frac{16}{4} \right) = 6.02 \text{ dB} \quad (2)$$

To make fair comparisons between different trellis codes, one needs to take into account both the error coefficient and the complexity of the code. Originally, the number of states (2^v) was used to determine the realization expense of the Viterbi decoder. Later, Wei proposed in [4] the ratio of the total number of allowed state transitions of a code to the number of signaling intervals, N_T , as a measure of code complexity, β . Our code has a low complexity $\beta = 32/4 = 8$.

The error coefficient normalized per two dimensions, N_o , is the average number of sequence paths at MSSED (N_{free}), divided by the number of 2D signaling intervals (N_T). For the code presented in this paper we have 14 paths of length one (parallel transitions) and 16 paths of length two at d_{free} ; the coded normalized error coefficient is therefore, $N_{o,c} = (14+16)/4 = 7.5$, while the normalized error coefficient of the base uncoded constellation is $N_{o,u} = 8/4 = 2$.

We use Forney's rule that every doubling of the error coefficient with respect to that of the uncoded system causes a loss of 0.2 dB [6]. The losses, λ , of our system are therefore

$$\lambda = \frac{\log \left(\frac{N_{o,c}}{N_{o,u}} \right)}{\log(32)} = \frac{\log \left(\frac{7.5}{2} \right)}{\log(32)} = 0.38 \text{ dB} \quad (3)$$

The effective gain of the X8-TCM scheme is thus $\gamma_{eff} = G_a - \lambda = 6.02 - 0.38 = 5.64$ dB. The error coefficient can be reduced by increasing the number of states on the trellis, thus trading complexity for a higher effective gain. The redundancy of X8-TCM is 1 bit per 8D, which yields a normalized redundancy per 2D, ρ , of 1/4.

Table I summarizes the characteristics of the TCM scheme presented in this paper and other TCM systems of rate 2 bit/T. The reference uncoded system is QPSK over N_T signaling intervals, except where indicated. C_c denotes the constellation or modulation used for the coded system. It is clear from table I that our system achieves a large asymptotic gain G_a , a high effective gain γ_{eff} and quite low complexity β and therefore outperforms most other schemes. The other two systems with performance equivalent to that of our system are coded Q²PSK with 16 states and phase/frequency TCM with 128 states. Both these systems have a much higher complexity, and in the case of phase/frequency TCM a slightly different bandwidth [9]. Some more conventional TCM systems with a large number of states (128 or more) achieve a performance slightly inferior to that of the system presented here.

In general, those codes with asymptotic and effective gain close to that of X8 usually suffer from one or more of the following: very high complexity, large constituent 2D constellations, high redundancy, constellation asymmetry or large number of nearest neighbours;

Our X8-TCM system avoids all these disadvantages while yielding large gains and maintaining a constant envelope, all with only eight trellis states.

VII. SUMMARY AND SUGGESTIONS

A new 8D TCM scheme, called X8-TCM, which uses a constellation of 512 points with unit energy per dimension in combination with a simple convolutional encoder with eight states and a QPSK modulator over four signaling intervals was presented in this paper. It has been shown that the increase in MSSED between coded sequences using this scheme yields an asymptotic gain larger than those of most previously published codes, without sacrificing data rate or increasing bandwidth. With the new scheme there is no constellation expansion penalty paid due to the efficient use of the 8D space. Also, the normalized redundancy is only 1/4 and the complexity is only 8. The error coefficient is 7.5 yielding an effective coding gain of 5.64 dB. The X8-TCM also has constant envelope and is invariant under rotations of 90°, 180° and 270°.

Simulations could be performed to more accurately evaluate the actual performance of the system presented, in terms of probability of error vs. SNR. Constellations similar to the one used here can be formed in other dimensions and used to design new TCM schemes with high gains.

VIII. REFERENCES

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TABLE I
Comparison of TCM schemes of rate 2 bit/T.

N	N_T	C_C	ν	R_C	ρ	G_a (dB)	N_0	β	$\gamma_{eff}^{(1)}$	Ref
1	1	8-PAM ⁽²⁾	2	1/2	2	3.31	8	8	3.11	[12]
1	1	8-PAM ⁽²⁾	3	1/2	2	3.77	8	16	3.57	[12]
1	1	8-PAM ⁽²⁾	4	1/2	2	4.18	16	32	3.78	[12]
1	1	8-PAM ⁽²⁾	7	1/2	2	5.81	132	256	4.80	[12]
1	1	As8-PAM ^(2,3)	2	2/3	2	3.55		16		[10]
1	1	As8-PAM ^(2,3)	3	2/3	2	4.15		32		[10]
2	1	8-PSK	2	1/2	1	3.01	1	8	3.01	[1]
2	1	8-PSK	3	2/3	1	3.60	2	32	3.60	[1]
2	1	8-PSK	4	2/3	1	4.13	2.3	64	4.09	[12]
2	1	8-PSK	7	2/3	1	5.17	0.5	512	5.17	[12]
2	1	8AM-PM	2	1/2	1	2.00		8		[1]
2	1	8AM-PM	3	2/3	1	3.01		32		[1]
2	1	8AM-PM	4	2/3	1	3.80		64		[1]
2	1	8AM-PM	7	2/3	1	5.10		512		[1]
2	1	Asy 8-PSK	4	2/3	1	4.15		64		[10]
4	1	Q ² PSK ⁽⁴⁾	2	3/4	1/2	4.77	2	8	4.77	[3]
4	1	Q ² PSK ⁽⁴⁾	3	3/3	0	3.01	2	32	3.01	[3]
4	1	Q ² PSK ⁽⁴⁾	4	1/2	1/2	6.02	2.3	64	5.98	[3]
4	1	2FSK/QPSK	2	1/2	1/2	3.01		8		[9]
4	1	2FSK/QPSK	3	2/3	1/2	3.7-4.8 ⁽⁵⁾		32		[9]
4	1	2FSK/QPSK	4	2/3	1/2	4.4-4.8 ⁽⁵⁾		64		[9]
4	1	2FSK/QPSK	7	2/3	1/2	5.4-6.0 ⁽⁵⁾		512		[9]
4	2	8-PSK	2	1/2	1/2	3.01	3	4	2.89	[7]
4	2	8-PSK	3	2/3	1/2	3.01	1	16	3.01	[7]
4	2	8-PSK	4	3/4	1/2	4.13	4	64	3.93	[7]
4	2	8-PSK	7	3/4	1/2	5.75	12.5	512	5.22	[7]
6	3	8-PSK	2	1/2	1/3	3.01	5	3	2.75	[7]
6	3	8-PSK	3	2/3	1/3	3.01	2.3	11	2.97	[7]
6	3	8-PSK	4	2/3	1/3	3.01	1	21	3.01	[7]
6	3	8-PSK	6	3/4	1/3	4.77	3.67	171	4.60	[7]
8	4	8-PSK	2	2/3	1/4	3.01	3	4	2.89	[7]
8	4	8-PSK	3	3/4	1/4	3.01	1	16	3.01	[7]
8	4	8-PSK	4	4/5	1/4	3.70	2	64	3.70	[7]
8	4	8-PSK	6	4/5	1/4	5.46	6	256	5.14	[7]
8	4	X8	3	2/3	1/4	6.02	7.5	8	5.64	Here

- (1) This quantity is obtained either by using the rule of 0.2 dB loss every doubling of N_0 or by curves given in reference for SNR at pe of 10^{-6} .
- (2) The reference system is uncoded 4-PAM.
- (4) The reference system is uncoded Q²PSK.
- (3) As means asymmetric.
- (5) Gains are dependent on the frequency modulation index h. Bandwidth is slightly changed.