# THE PERFORMANCE OF DUALLY POLARIZED M-QAM AND L-QPRS SYSTEMS WITH CROSSTALK AND DIFFERENTIAL PHASE SHIFT

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## ABSTRACT

The probability of symbol error ($P_E$) performance is studied for cross-polarized M-QAM and L-QPRS systems operating in the depolarization crosstalk and differential phase shift environment. Explicit general formulas are provided and results presented for dual channel 4-QAM (QPSK) and 9-QPRS. It is demonstrated that the $P_E$ varies with a period of 90° with differential phase shift and that L-QPRS systems are less sensitive to differential phase shift than the corresponding M-QAM system; the sensitivity is measured by the ratio of maximum to minimum $P_E$.

## I. INTRODUCTION

Digital communication systems can have increased route capacity by utilizing dual orthogonal polarizations of two transmitted information-bearing signals on the same carrier frequency. These systems will then be susceptible to co-channel interference due to atmospheric propagation impairments and antenna imperfections. Two bandwidth efficient digital modulation schemes that have attracted the attention of many researchers are M-ary Quadrature Amplitude Modulation (M-QAM) and L-ary Quadrature Partial Response Signals (L-QPRS), whose performance we study in this paper.

Prabhu [1,2] and Borgne [3] used upper bounds for the probability of error for M-QAM systems, assuming that the co-channel interference was totally incoherent with the desired channel. Duvoisin et al. [4] assumed that the desired and interfering signals originated from the same coherent source and described a method that gives exact results for M-QAM. Their method was extended to include differential phase shift [5]. The performance of 225-QPRS has been determined by computer simulation by Wu and Feher [6].

In this paper, a method [7,8] which gives exact results for the probability of error ($P_E$) for "square" M-QAM and a tight upper bound for precoded L-QPRS is described. It is then shown that L-QPRS systems are less sensitive to differential phase shift than the corresponding M-QAM system and that the $P_E$ varies with a period of 90° with differential phase shift.

## II. MODEL OF THE DUAL-POLARIZED M-QAM AND L-QPRS SYSTEMS

The information signals at sample time n are written as

$$I_i(n) = a_i(n) + jb_i(n), \quad i = 1,\ 2 \tag{1}$$

where $a_i(n)$ and $b_i(n)$ can take on any value in the set $\{0, \pm 2, \pm 4, \ldots, \pm L^{1/2}-1\}$ with probabilities

$$P[a_i(n)=2p] = P[b_i(n)=2p] = \frac{J-|p|}{J^2} \tag{2}$$

for L-QPRS, and any value in the set $\{\pm 1, \pm 3, \ldots, \pm M^{1/2}-1\}$ with probabilities

$$P[a_i(n)=2s-1-M^{1/2}] = P[b_i(n)=2s-1-M^{1/2}p] = M^{-1/2} \tag{3}$$

for M-QAM, where $p = 0, \pm 1, \pm 2, \ldots, \pm(L^{1/2}-1)/2$; $s=1,2, \ldots, M^{1/2}$; and $J = (L^{1/2}+1)/2$.

The channel is assumed to be slowly varying and non-dispersive, and to corrupt the transmission by introducing a fraction of one information stream into the other and by introducing AWGN. Thus, the model is applicable to the satellite path, which takes on new significance for multi-level constellations, since it has recently been demonstrated by Feher [9] and Cheung and Aghvami [10] that 16-QAM systems are possible over the satellite path.

The received signals are passed through the receiver filter and sampled every T seconds. For M-QAM, raised-cosine filtering is assumed, and for L-QPRS duobinary filtering is assumed, with equal filtering at the transmitter and receiver to ensure optimum performance [11]. After the filtering operation, we get

$$Y_i(n) = \sum_{j=1}^{2} T_{ij}\, I_j(n) + N_i(n), \quad i = 1,2 \tag{4}$$

The factors $T_{11}$ and $T_{22}$ account for direct path attenuation and phase shift, and $T_{12}$ and $T_{21}$ account for the depolarization crosstalk.. In this paper we assume the following conventional approximations to the satellite path [12]:

$$\begin{aligned} T_{11} &= T_{22} = 1 \\ T_{12} &= k\exp(j\phi_1) \\ T_{21} &= k\exp(j\phi_2) \end{aligned} \tag{5}$$

where $\phi_i$, $i=1,2$ is the differential phase shift from channel i. The sequences $N_1(n)$ and $N_2(n)$ are independent samples of zero mean, complex valued Gaussian processes with equal variances $2\sigma^2$, with $\sigma^2 = 2N_o/\pi$ for L-QPRS and $\sigma^2 = N_o/2$ for M-QAM, where $N_o$ is the double-sided spectral density of a single complex noise source. The average transmitted symbol energy, $E_s$, in each channel is $8(J^2-1)/3\pi$ for L-QPRS and $2(M-1)/3$ for M-QAM. We define the signal-to-noise ratio (SNR) as $E_s/N_o$.

## III. PROBABILITY OF SYMBOL ERROR

In this section, the probability of symbol error ($P_E$) of one channel will be determined. Substituting (5) in (4) gives

$$Y_1(n) = I_1(n) + k\exp(j\phi_1)I_2(n) + N_1(n) \tag{6}$$

Here, $I_1(n)$ is the desired signal and $I_2(n)$ is the interfering signal, whose constellation is assumed to be square and quadrantly symmetric, that is , $a_2$, $-a_2$, $b_2$ and

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$-b_2$ take on the same values with the same probabilities, and $P(a_2,b_2) = P(a_2) . P(b_2)$ if $P(a_2,b_2) \neq 0$.

The in-phase and quadrature-phase components of $Y_1$ are

$$X = \mathrm{Re}(Y_1) = a_1 + k(a_2\cos\phi_1 - b_2\sin\phi_1) + N_{1R}$$

$$= a_1 + c_x + N_{1R} \qquad (7)$$

$$Y = \mathrm{Im}(Y_1) = b_1 + k(a_2\sin\phi_1 + b_2\cos\phi_1) + N_{1I}$$

$$= b_1 + c_y + N_{1I} \qquad (8)$$

Note that X and Y are independent Gaussian variables, with means of $a_1 + c_x$ and $b_1 + c_y$, respectively, and equal variances $\sigma^2$. The probability of symbol error and its expected value, respectively, can be written as

$$P_E = P_{EI} + P_{EQ} - P_{EI}P_{EQ} \qquad \text{and}$$

$$E(P_E) = E(P_{EI}) + E(P_{EQ}) - E(P_{EI}P_{EQ}) \qquad (9)$$

where $P_{EI}$ is the probability of symbol error in the in-phase component and $P_{EQ}$ is the probability of symbol error in the quadrature-phase component, and the expectation operator E(.) is with respect to the variables $c_x$ and $c_y$.

For M-QAM, it can be easily shown that

$$P_{EI} = (1/2)(1-M^{-1/2})[\mathrm{erfc}((1-c_x)/N)$$

$$+ \mathrm{erfc}((1+c_x)/N)] \qquad (10)$$

where $N = 2^{1/2}\sigma$. Also, $P_{EQ} = P_{EI}|x{=}y$, if $c_x$ is replaced by $c_y$.

Since $c_x$, $-c_x$, $c_y$ and $-c_y$ take on the same values with the same probabilities, it follows that

$$E(P_EI) = E(P_EQ) = (1-M^{-1/2})\ E[\mathrm{erfc}((1+c_x)/N)] \qquad (11)$$

Also for quadrantly symmetric constellations, it is straightforward to show that [7]

$$E(P_{EI}P_{EQ}) =$$

$$(1-M^{-1/2})^2E[\mathrm{erfc}((1-c_x)/N)\mathrm{erfc}((1-c_y)/N)] \qquad (12)$$

So, the $P_E$ for square M-QAM is given by

$$E(P_E) = 2E(P_{EQ}) - E(P_{EI}P_{EQ}) \qquad (13)$$

Using (10)-(13), explicit formulas for the probability of symbol error can be obtained.

Likewise, for L-QPRS, if the boundaries of the decision regions are assumed to be independent of the SNR, and are therefore located halfway between the symbols as in M-QAM, then it is easily shown [8] that (10)-(13) still apply, provided $M^{1/2}$ is replaced by $J^2$. Also, (10)-(13) result when no consideration is given to the fact that for L-QPRS a symbol can be outside its decision region without an error being committed [13]. Thus, (10)-(13) are to be viewed as tight upper bounds for L-QPRS systems.

## IV. AVERAGE PROBABILITY OF SYMBOL ERROR FOR L-QPRS AND M-QAM

The probability of symbol error has been determined for L-QPRS and M-QAM with the differential phase shift $\phi_1$ as a parameter. To remove this dependence on $\phi_1$, the usual procedure is to assume that $\phi_1$ is uniformly distributed from $-\pi$ to $\pi$, and an average value is taken. When this is done, the result is

$$\langle P_E \rangle = 1/(2\pi) \int_{-\pi}^{\pi} P_E \, d\phi_1 \qquad (14)$$

## V. RESULTS AND DISCUSSION

Figure 1 shows $\mathrm{Log}_{10}(\langle P_E \rangle)$ versus SNR(dB) with the crosstalk level k (dB) as a parameter for dual channel 4-QAM and 9-QPRS. As expected, these plots show that the average probability of symbol error increases with the crosstalk level, and that 4-QAM performs better than 9-QPRS.

Plots of $P_E$ vs. differential phase for the systems studied here show a 90° periodic variation (approximately sinusoidal in shape) of $P_E$ with differential phase, which is intuitively reasonable since the constellations remain invariant under a 90° rotation. For low levels of crosstalk and medium to high SNR the minimum $P_E$ is at $\phi_1=0°$, and the maximum is at $\phi_1= 45°$. However, for low SNR the minimum occurs at $\phi_1=45°$, and the maximum at $\phi_1=0°$. Figure 2 illustrates this behavior for 9-QPRS.

The differential phase shift usually varies slowly [14] compared with the data rate. Therefore, if the differential phase shift is uniformly distributed from $-\pi$ to $\pi$, there will be relatively long times when the $P_E$ is at its minimum value and relatively long times at which it is at its maximum. It might therefore be possible for adaptive systems which can measure differential phase shift to take advantage of this effect. Systems with marginal signal-to-noise ratios will be fine for differential phase shifts around zero degrees, but may not perform well when it shifts to 45°. A measure of the sensitivity to differential phase shift is given by the ratio of the maximum $P_E$ to minimum $P_E$. This was done for the two systems described here and plots of the logarithm of the ratio of $P_{Emax}$ to $P_{Emin}$ vs. SNR (dB) with the crosstalk level as a parameter can be found in Fig. 3. From this figure it is seen that the ratio increases with increasing crosstalk level and increasing SNR if the crosstalk level is not increased beyond a certain level, after which the ratio actually decreases with increasing level. (This can be seen by inspection of the -10 dB and -15 dB curves for 9-QPRS, but it is also true for QPSK.)

Another important observation is that the ratio of $P_{Emax}$ to $P_{Emin}$ is less for L-QPRS than it is for the corresponding M-QAM system, for a given SNR and crosstalk level. For example, for SNR = 16dB and k = -10dB, $\log(P_{Emax}/P_{Emin}) = 1.2$ for QPSK, and for 9-QPRS it is 0.4. This can be explained with the help of Fig. 4, which shows the constellation and decision regions for the desired channel for 9-QPRS along with the possible dislocations within a decision region due to the crosstalk. As can be seen from Fig. 4(b) , the minimum distance is strongly dependent upon the differential phase shift. However, because the minimum distance $d_{min}$ occurs less frequently $(P(d_{min})=1/J^2)$ , the probability of symbol error does not depend upon $d_{min}$ as much as it does for the M-QAM case. Hence, the probability of symbol error is not as sensitive to differential phase shift for L-QPRS as it is for M-QAM systems. This implies that if a Viterbi decoder is used to recover the SNR degradation of L-QPRS, then L-QPRS should have a better performance in the crosstalk environment than the corresponding M-QAM system.

## VI. CONCLUSIONS

In this paper explicit formulas have been given for probability and average probability of error performance for dual-polarized systems, where (i) the desired channel has a square M-QAM constellation and the interfering channel has a constellation which is quadrantly symmetric, and (ii) the desired channel has a L-QPRS constellation and the interfering channel's constellation is quadrantly symmetric. In the latter case, the explicit formulas provided are a tight upper bound on the system's performance.

It has been established that L-QPRS systems are less sensitive to differential phase than M-QAM systems, and will perform better under certain crosstalk conditions including differential phase shift than the corresponding M-QAM system, particularly if Viterbi decoding is used.

The probability of symbol error was also shown to vary periodically with differential phase, with a period of 90°. This periodic variation appears to have a large dominant sinusoid at the fundamental frequency. However, for large crosstalk values, it also contains large harmonics.

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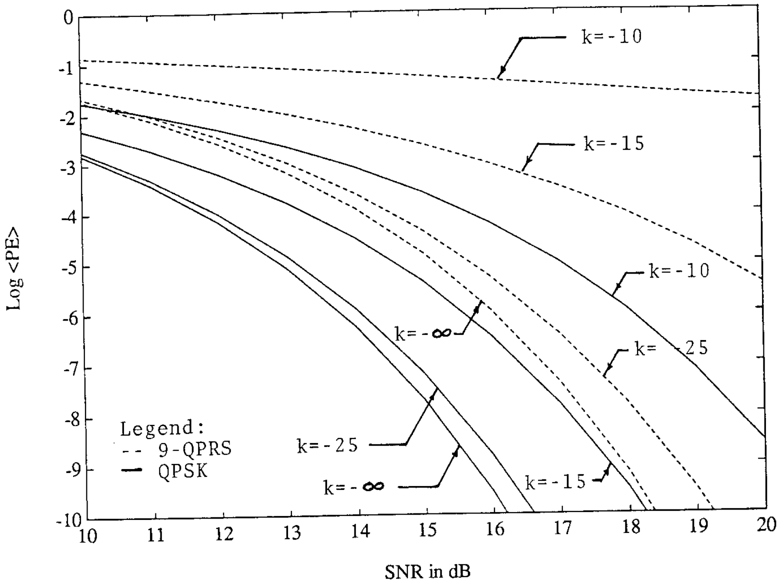

Fig. 1. Log $\langle P_E \rangle$ vs. SNR (dB) with crosstalk level k (dB) as a parameter for dual-channel QPSK and 9-QPRS.

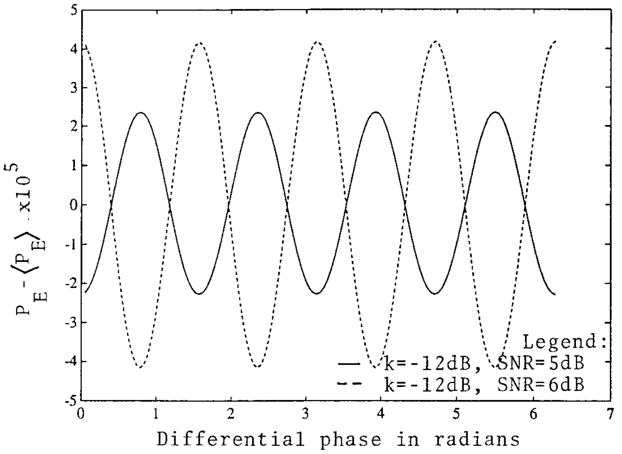

Fig. 2. Variation of probability of error vs. differential phase shift for 9-QPRS for crosstalk level of -12 dB and SNR of 5 and 6 dB.

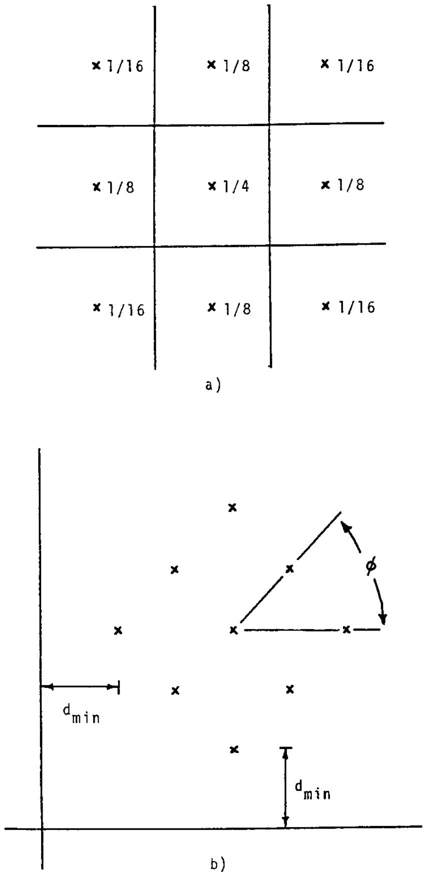

Fig. 4. a) Desired signal constellation and decision boundaries.
b) Possible dislocations within a decision region due to crosstalk for 9-QPRS.

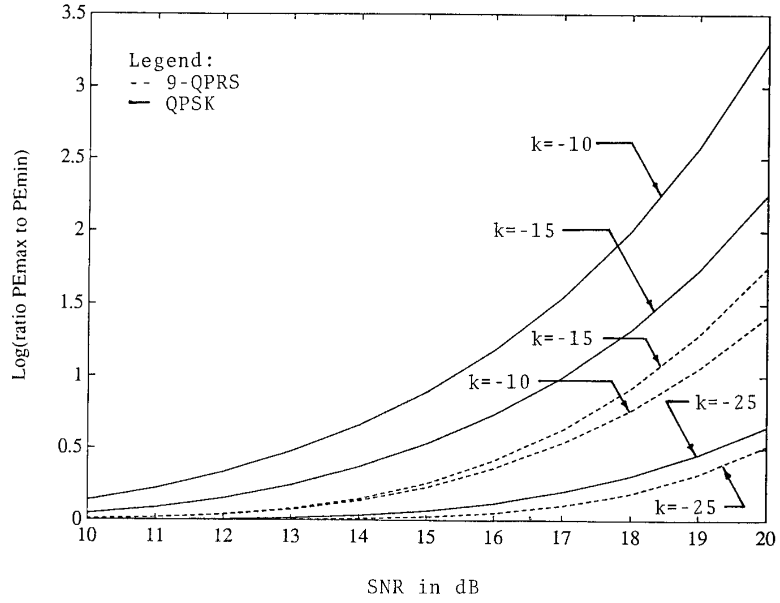

Fig. 3. Log of the ratio of max. $P_E$ to min. $P_E$ vs. SNR (dB) for dual-channel 9-QPRS and QPSK. The crosstalk level k (dB) is a parameter.