Further Results Related to Power Supply Design and Analysis in the Undergraduate Curriculum

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***Abstract*—The power supply analysis and design results of Sherman and Hamacher are extended to include approximate solutions to the nonlinear equations that define the rectifier turn-on angles. Additionally, peak rectifier current solely as a function of ripple factor, as well as the inverse relationship, i.e., ripple factor as a function of peak rectifier current, are also provided. Simplified exact equations and also new approximations are given for rms transformer current. These new approximations allow the ripple factor to be determined as a function of rms transformer current. These results are for both the constant current and resistive load supplies.**

***Index Terms*—Nonlinear equation approximation, power supplies, power supply design.**

I. Introduction

Recently, Sherman and Hamacher [1] have presented new results for the design and analysis of power supplies, both for the resistive load power supply (RLP) and for the constant current load power supply (CCLP). Their paper goes a long way in enabling more meaningful design and analysis of power supplies in the undergraduate curriculum, especially when the ripple is not small. Notwithstanding the substantial contribution of their paper, there remains opportunity to extend the results presented.

The purpose of this paper is to extend the results of Sherman and Hamacher by providing the following, for both types of supply:

1) Approximate solutions to the nonlinear equations derived by Sherman and Hamacher that define the rectifier turn-on angles. (Sherman and Hamacher provided approximate solutions only for the turn-off angles.) This means that students will not need a programmable calculator to solve the nonlinear equations, but more importantly, students will be able to see explicit relationships between variables.
2) The peak rectifier current is given solely in terms of the ripple factor, so that the role of the ripple factor is clearly seen. Additionally, for RLP, the peak rectifier current is determined for ripple factors greater than 0.57. (Sherman and Hamacher [1] provided results only for ripple factors less than 0.57.)
3) The ripple factor is determined as a function of peak rectifier current and root mean square (rms) transformer current. This is of value when one has in mind a particular rectifier and a particular transformer, and wishes to investigate the possibility of their use in a given design, and hence needs to determine the minimum allowable ripple factor.
4) Simplified exact equations are given for the rms current in the transformer, as well as new approximations, which allow the ripple factor to be determined from the rms transformer current.

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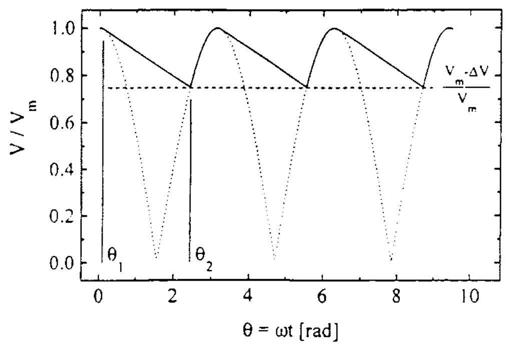

Fig. 1. Filter capacitor voltage for constant current load for a ripple of 25%. This figure is taken from [1].

II. Preliminaries

Fig. 1 shows the full-wave rectified waveform, where the cutoff angle θ_1 and the angle at which the rectifier begins to conduct, θ_2, are defined. It should be noted that the rectifier conducts for a total of $\pi + \theta_1 - \theta_2$, and hence is nonconducting for $\theta_2 - \theta_1$. Sherman and Hamacher [1] have shown that the CCLP must satisfy the following equation:

$$\cos\theta_1 + \cos\theta_2 - (\theta_2 - \theta_1)\sin\theta_1 = 0 \tag{1}$$

where

$$\cos\theta_2 = r - 1, \quad \text{or} \quad \cos(\pi - \theta_2) = 1 - r \tag{2}$$

with r as the ripple factor and is equal to $\Delta V/V_m$, where $\Delta V = V_m - V_r$, V_r is the voltage at angle θ_2, and V_m is the maximum voltage.

For the RLP, Sherman and Hamacher [1] give the following equation to be satisfied:

$$\cos\theta_2 + \cos\theta_1 \exp[-(\theta_2 - \theta_1)\tan\theta_1] = 0. \tag{3}$$

It has been the practice to assume that for small ripple, the exponential voltage decay of the resistive load can be approximated by a straight line decay, i.e., the same decay as the constant current load. Therefore, it is reassuring that (3) reduces to

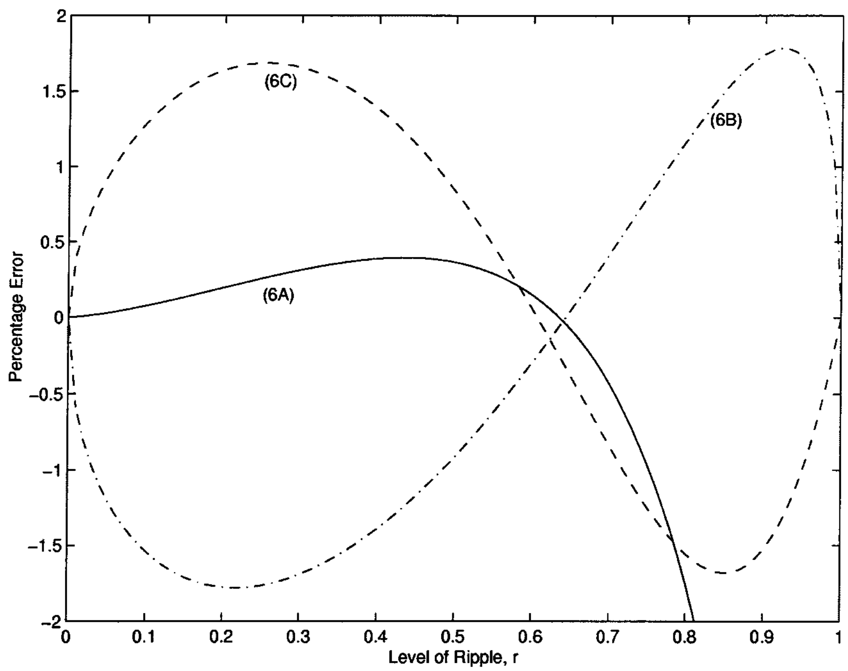

Fig. 2. Percentage error of approximations for the turn-on angle θ_2, of the constant current supply, (6A)-[i.e., (6) with $k_1 = 1, k_2 = 1$], and (6B)-[i.e., (6) with $k_1 = 1.6114, k_2 = 1.2893$], and for the resistive load supply, (6C)-[i.e., (6) with $k_1 = 1.0095, k_2 = 0.835\,15$].

(1) for small ripple. This is easily demonstrated by replacing the exponential function in (3) by the first two terms of its Taylor series expansion. As a consequence of the above, the solution for (1) and (3) must coincide for the small ripple case.

III. Approximate Solutions of the Defining Equations

A. Approximate Solutions of the Defining Equation for for the Constant Current Load Power Supply (CCLP)

In the design phase, a required ripple factor is usually given, and hence the solution of (1) will give θ_1 in terms r and θ_2, which of course are related by (2). Indeed, [1, eq. (12)] provides an excellent approximation (accurate to 0.5% for $0 \le r \le 0.8$) that can be written as

$$\sin\theta_1 = \frac{r}{\theta_2(1 - 0.07r - 0.195r^3)}. \tag{4}$$

Knowing $\sin\theta_1$ from (4) allows the capacitor value to be determined from (5) of [1], i.e.,

$$\sin\theta_1 = \frac{I_{\rm dc}}{\omega C V_m} \tag{5}$$

where $I_{\rm dc}$ is the constant discharge current of the filter capacitor C (which is also assumed to be the value of the load current), and ω is the radian frequency of the applied ac voltage.

After the power supply has been designed, it is sometimes necessary to analyze its performance, i.e., to determine for example what ripple level r has been achieved. In order to do this, (1) has to be solved for θ_2 as θ_1 is now known from (5).

There are many ways of deriving an approximate solution to (1), given the value of θ_1. The best approximation that the author has found is derived by first noting that a solution to (1) is given by $\theta_2 = r/\theta_1$, for small r. This can be seen by approximating $\cos\theta_1$ by 1 and using $\sin\theta_1 \approx \theta_1$. Furthermore, from (2), $\theta_2 \approx \pi - \sqrt{2r}$. Hence, $\theta_1 = \pi - \sqrt{2\theta_1\theta_2}$. Solving this equation for θ_2 gives the desired result

$$\theta_2 = \pi + k_1\theta_1 - \sqrt{(\pi + k_2\theta_1)^2 - \pi^2} \tag{6}$$

where $k_1 = 1$ and $k_2 = 1$.

The percentage error performance for (6) with $k_1 = 1$ and $k_2 = 1$, is shown in the curve (6A) of Fig. 2, which clearly shows that (6) is indeed an excellent approximation to the turn on angle for $r \le 0.8$.

However, an approximation that is valid for all values of r can be obtained if one sets $k_2 = 2[\sqrt{((1 + k_1)/2)^2 + 1} - 1]$, and k_1 is optimized to give near minimax percentage error performance. Indeed, $k_1 = 1.6114$ gives percentage errors within $\pm 1.8\%$, as shown in the curve (6B) of Fig. 2.

The relationship between k_1 and k_2 is established by noting that for $r = 1, \theta_2 = \theta_1 = (\pi/2)$, and these are then substituted into (6).

After θ_2 is found, the level of ripple can then be determined from (2).

B. Approximate Solutions of the Defining Equation for the Resistive Load Power Supply (RLP)

For the design phase for the RLP, (3) has to be solved for θ_1 in terms of θ_2 or r. In fact, [1, eq. (20)] can be used as it provides an excellent approximation (accurate to 0.5% for $0 \le r \le 0.8$) and can be written as

$$\tan\theta_1 = \frac{r}{\theta_2(1 - 0.586r - 0.358r^3)}. \tag{7}$$

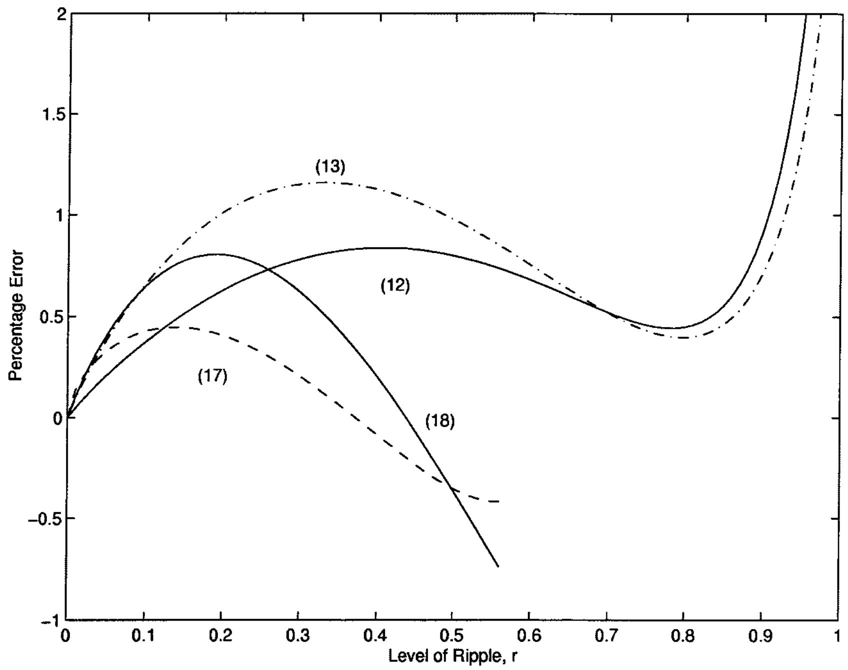

Fig. 3. Percentage error of approximations for (a) the peak rectifier current as a function of ripple factor for (i) the constant current supply, (12), (ii) the resistive load supply, (17), and (b) the ripple factor as a function of peak rectifier current for (i) the constant current supply, (13), (ii) the resistive load supply, (18). Please note that (17) and (18) are only valid for $r \leq 0.57$.

Knowing $\tan\theta_1$ from (7) allows the capacitor value to be determined from (18) of [1], i.e.,

$$\tan\theta_1 = \frac{1}{\omega RC} \tag{8}$$

where R is the load resistor.

Notwithstanding the excellent approximation of (7), the simple approximation in (9) below is of value in deriving (10), which in turn is used to derive (16) and (18).

$$\theta_1 = \frac{2r}{(2-r)\theta_2}. \tag{9}$$

Equation (9) is derived by approximating $\cos\theta_1$ by 1 and using $\tan\theta_1 \approx \theta_1$, along with $e^{-x} \approx (2-x)/(2+x)$ in (3). Furthermore, using $\cos x \approx (4-x^2)/(4+x^2)$ in (2) gives $\theta_2 \approx \pi - \sqrt{(4r)/(2-r)}$. Hence, (9) becomes

$$\theta_1 = \frac{2r}{\sqrt{2-r}(\pi\sqrt{2-r} - 2\sqrt{r})}. \tag{10}$$

In the analysis case, (3) has to be solved for θ_2 in terms of θ_1, which is now found from (8). Fortunately, (6) can be used in this case, as well, but now with $k_1 = 1.0095$, and the corresponding k_2 value determined with the relationship established in the previous section. These values give percentage errors within ±1.7%, as is shown in curve (6C) of Fig. 2.

IV. Determination of the Peak Rectifier Current

A. CCLP

In the design of power supplies, it is necessary to know the peak value of the rectifier current since this determines one of the specifications the rectifier must satisfy. Sherman and Hamacher have provided an expression of this for the constant current supply as

$$i_{\text{peak}} = I_{\text{dc}}\left(1 + \frac{\sin\theta_2}{\sin\theta_1}\right). \tag{11}$$

The disadvantage of the expression (11) is that it does not directly relate the peak current to the desired level of ripple. Furthermore, as pointed out by the authors, the value of i_{peak} becomes significant only for small values of ripple. Hence, a small ripple approximation to (11) would be useful and should be relatively easy to find. Indeed, from (2), one finds $\sin\theta_2 = \sqrt{r(2-r)}$. Substituting this into the numerator of (11), using $\sin\theta_1 \approx \theta_1$ and $\theta_1 \approx r(\pi - \sqrt{(4r)/(2-r)})^{-1}$ gives

$$\frac{i_{\text{peak}}}{I_{\text{dc}}} = \pi\sqrt{\frac{2-r}{r}} - 1. \tag{12}$$

It is rewarding to see the excellent accuracy that (12) provides as shown in Fig. 3.

A further advantage to (12) is that it is easily inverted to solve for r. This would be helpful when one has already selected a rectifier with a given peak current value, and wants to know the minimum ripple factor that the diode could tolerate.

Inverting (12) gives

$$r = \frac{2\pi^2}{\left(\frac{i_{\text{peak}}}{I_{\text{dc}}} + 1\right)^2 + \pi^2}. \tag{13}$$

The percentage error performance of (13) can also be seen in Fig. 3.

B. Resistive Load (RLP)

In the resistive load power supply, Sherman and Hamacher gives the rectifier current in [1, eq. (21)]. However, this can be written as

$$\frac{i}{I_m} = \frac{\sin(\theta - \theta_1)}{\sin\theta_1} \quad \text{for } \theta_2 < \theta \leq \pi + \theta_1 \tag{14}$$

where $I_m = V_m/R$.

The upper limit $\pi + \theta_1$ is valid because of the periodicity of the current.

It is further stated that the peak current will occur at $\theta = \theta_2$, for r values less than 0.57, but no statement is made concerning the peak value for r values greater than 0.57. However, it is now possible to show that this peak value is given by

$$\frac{i_{\text{peak}}}{I_m} = \frac{1}{\sin\theta_1}. \tag{15}$$

To see this, note that (14) is a maximum when the numerator is a maximum, i.e., when $\theta = (\pi/2) + \theta_1$, which results in (15). However, θ must be greater than θ_2 (since the current is zero for $\theta_1 \leq \theta \leq \theta_2$). Hence, the maximum occurs at θ_2 when $\theta_2 > (\pi/2) + \theta_1$.

To find the peak current in terms of r, set $\theta = \theta_2$ in (14), expand the compound angle, and then substitute (2) and $\sin\theta_2 = \sqrt{r(2-r)}$. Furthermore, substitute (10) for θ_1 and use the approximation $\tan\theta_1 \approx \theta_1$ to give:

$$\frac{i_{\text{peak}}}{I_m} = \sqrt{\frac{2-r}{r}}\left[\pi\left(\frac{2-r}{2}\right) - \sqrt{r(2-r)}\right] + 1 - r. \tag{16}$$

However, recall that (16) is only valid for r values less than 0.57, and for this range, the author has found that a better approximation is given by

$$\frac{i_{\text{peak}}}{I_m} = \sqrt{\frac{2-r}{r}}\left[\pi\left(\frac{2-r}{2}\right) - 0.979\sqrt{2r}\right] + 1 - r. \tag{17}$$

The percentage error of (17) is shown in Fig. 3, where it is clear that (17) is an excellent approximation.

To find the peak rectifier current in terms of r solely, for r greater than 0.57, solve (2) for θ_2, then use this to solve (7) for θ_1 and substitute into (15).

Unfortunately, (17) is not easily inverted in closed form, if it is desired to find r in terms of the peak rectifier current. Nevertheless, Appendix I shows how this expression can be found, and is given in (18) below.

$$r = -1.0219 + \sqrt{1.0442 + 2.0437\left[2 + I_p^2 - \sqrt{\left(2 + I_p^2\right)^2 - 4}\right]} \tag{18}$$

where $I_p = ((i_{\text{peak}}/I_m) + 1)/(\pi)$.

Please note that (18) is valid for r values less than 0.571 28, or $(i_{\text{peak}}/I_m) \geq 2.33$.

To determine r values for $i_{\text{peak}}/I_m < 2.33$, (15) could be solved for θ_1, and r determined from (10).

C. Designing for Peak Rectifier Current

Given a desired peak rectifier current for CCLP, (13) [(18) for RLP] can be used to find the ripple factor r, which can then be substituted into (2) to obtain θ_2, and these in turn can be used in (4) and (5) [(7) and (8) for RLP] to obtain the values of $\sin\theta_1$ [$\tan\theta_1$ for RLP] and the value of the capacitor C, respectively. Of course, the closest commercial value of C will have to be used, and hence (5) [(8) for RLP] can be used to determine the actual value of θ_1, (6) is then used to find θ_2, and hence r with (2). Finally, (11) [(14) or (15) for RLP] is used to verify the desired peak rectifier current has been achieved.

V. Determination of the RMS Transformer Current

A. CCLP

Sherman and Hamacher have extended our understanding of power supply design by deriving an expression for the rms current in the transformer, which is given in [1, eq. (14)]. However, this expression can be simplified as will now be demonstrated. From (1), $\sin\theta_1$ can be found and substituted into the first term under the radical of [1, eq. (14)], thereby making that term equal to $\pi + \theta_2 - \theta_1$. Hence, the rms current I in the transformer will simplify to

$$I = \frac{I_{\text{dc}}}{2\sqrt{\pi}\sin\theta_1} \times \sqrt{4\pi - 2(\pi + \theta_2 - \theta_1)\cos 2\theta_1 + \sin 2\theta_2 - \sin 2\theta_1}. \tag{19}$$

Sherman and Hamacher also gives a very good approximation to (19) as

$$\frac{I}{I_{\text{dc}}} = \sqrt{\frac{3.96 - 1.37r}{\pi - \cos^{-1}(r-1)}}. \tag{20}$$

If one wanted to find the minimum ripple factor that could be tolerated by a given transformer, it would be necessary to have an expression of r as a function of I/I_{dc}. This is not easily accommodated by (20). However, (19) can be approximated by (21) below

$$\frac{I}{I_{\text{dc}}} = \sqrt{\frac{3.96 - 1.37r}{\sqrt{2r + .397r^2}}}. \tag{21}$$

Fig. 4 shows the percentage error of (21), which clearly shows that it is quite competitive to that of (20). In fact, (21) has lower percentage error than (20) for $0.16 \leq r \leq 0.82$. For $r < 0.16$, the percentage error of (21) is only slightly higher than that of (20), becoming equal as r goes to zero.

Inverting (21) produces

$$r = -\frac{Q^4 + 5.425}{.397Q^4 - 1.877} \pm \sqrt{\left(\frac{Q^4 + 5.425}{.397Q^4 - 1.877}\right)^2 + \frac{15.68}{.397Q^4 - 1.877}} \tag{22}$$

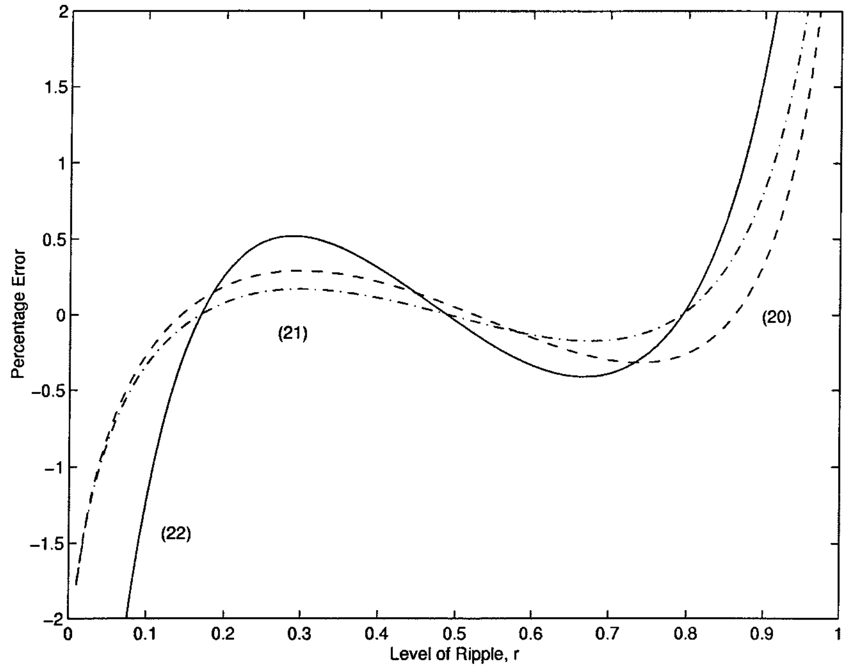

Fig. 4. Percentage error of approximations for the rms transformer current as a function of ripple factor, (20) and (21), and ripple factor as a function of rms transformer current, (22), for the constant current supply.

where $Q = I/I_{\rm dc}$, the positive sign is for $Q > 1.47$, and the negative sign is needed for $Q \leq 1.47$.

Figure 4 also shows the percentage error of (22), where it can be seen that it is excellent (less than 1%) for $0.11 \leq r \leq 0.87$. Furthermore, on inspection of Fig. 4, the steepness of the percentage error of (22) as r goes to zero might appear to be a problem. However, the percentage error at $r = 10^{-6}$ is less than 11%, and therefore (22) can be used with confidence for practical values of r.

B. Resistive Load (RLP)

For the RLP, the rms current I in the secondary winding of the transformer is a bit more involved than that of the constant current load. It is determined by the authors of [1] as $(I/I_{\rm dc}) = (I/I_m)(I_m/I_{\rm dc})$, where the ratio of $I_{\rm dc}/I_m$ is given by [1, eq. (25)], and a very good approximation of this by [1, eq. (26)], and the ratio of I/I_m by a complicated [1, eq. (27)]. Fortunately, this expression can be simplified to

$$\frac{I}{I_m} = \frac{1}{2\sqrt{\pi}\sin\theta_1}\sqrt{2(\pi+\theta_1-\theta_2)+\sin(2\theta_2-2\theta_1)}. \tag{23}$$

This follows by simply substituting (14) into the rms formula below

$$I = \sqrt{\frac{1}{\pi}\int_{\theta_2}^{\pi+\theta_1}\left(\frac{I_m\sin(\theta-\theta_1)}{\sin\theta_1}\right)^2 d\theta}. \tag{24}$$

Sherman and Hamacher also gives a very good approximation to the rms current I as

$$\frac{I}{I_{\rm dc}} = \sqrt{\frac{4.11-2.38r}{\pi-\cos^{-1}(r-1)}}. \tag{25}$$

As before, it is not easy to invert (25). However, (26) below can be used instead of (25). The advantage to (26) is that it is easily inverted

$$\frac{I}{I_{\rm dc}} = \sqrt{\frac{4.11-2.38r}{\sqrt{2r+.412r^2}}}. \tag{26}$$

Fig. 5 shows the percentage error of (26), which clearly shows that it is also quite competitive to that of (25), and therefore can be used in place of (25), without cause for concern.

Inverting (26) produces

$$r = -\frac{Q^4+9.78}{.412Q^4-5.664} \pm\sqrt{\left(\frac{Q^4+9.78}{.412Q^4-5.664}\right)^2+\frac{16.89}{.412Q^4-5.664}}, \tag{27}$$

where the positive sign is for $Q \geq 1.93$, and the negative sign is needed for $Q < 1.93$.

Fig. 5 also shows the percentage error of (27), where it can be seen that quite accurate estimates of the ripple factor as a function of rms transformer current can be achieved for all values of r less than about .95.

C. Designing for RMS Transformer Current

Given a desired rms transformer current for the CCLP, use (22) [(27) for RLP] to obtain r, then (2) for θ_2, followed by (4) and (5) [(7) and (8) for RLP] to obtain the value of C. Again the closest commercial value of C will have to be used, and thus (5) [(8) for RLP] can be used to determine the actual value of θ_1 achieved, (6) is then used to find θ_2, and hence r with (2). Finally, (20) [(25) for RLP] is used to verify the desired rms transformer current has been achieved.

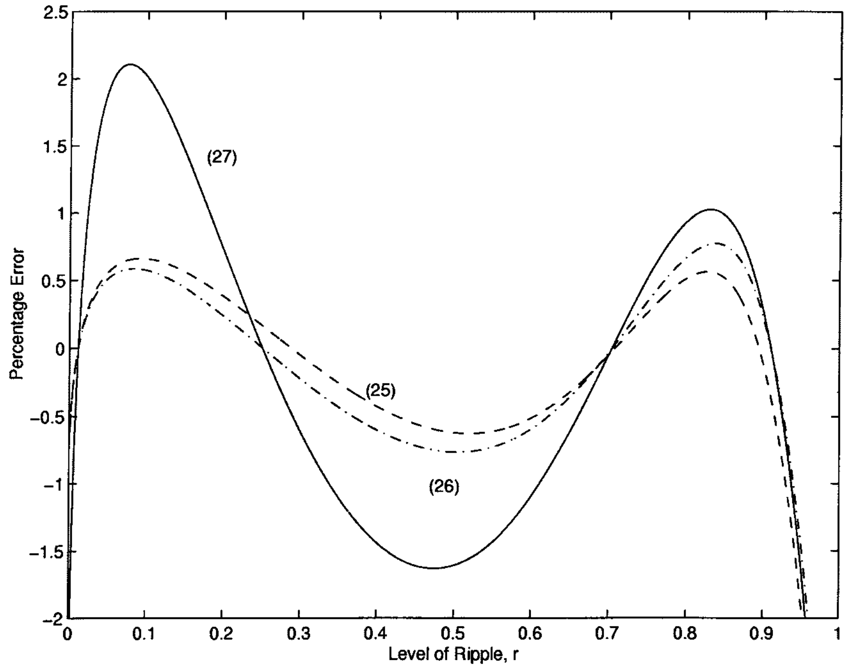

Fig. 5. Percentage error of approximations for the rms transformer current as a function of ripple factor, (25) and (26), and ripple factor as a function of rms transformer current, (27), for the resistive load supply.

VI. Conclusion

In this paper, new results have been derived which further extend knowledge of resistive load and constant current power supply design and analysis. Specifically, excellent approximate solutions were found to the nonlinear equations which allow the turn-on angles to be found in closed form. Furthermore, peak rectifier currents were determined in terms of ripple factor, and vice versa. Additionally, rms transformer currents were found in terms of ripple factor, and vice versa.

Appendix I

In this Appendix, (18) will be derived.

Recall that $\theta_2 \approx \pi - \sqrt{2r}$. Substitute this into (14) (setting $\theta = \theta_2$) and use $\sin\theta \approx \theta$ to get $\sqrt{2r} = ((i_{\mathrm{peak}}/I_m) - 1)\theta_1$. Furthermore, substitute (10) into the previous equation, and use $2\sqrt{r(2-r)} \approx \sqrt{8r}$ to produce re, an estimate for r, where $re = 2 + I_p^2 - \sqrt{[2+I_p^2]^2 - 4}$. Unfortunately, re has poor percentage error. In fact, the percentage error varies almost linearly with respect to the true value of r, and can be written as $100(re - r)/(r) = Kr$, where K is a constant. Let $k = K/100$ and solve to produce $r = -(1/2k) + \sqrt{(1/(2k))^2 + (re/k)}$. Optimizing k to give minimax error performance for all values of r less than 0.57 produces (18).

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