

Blind Phase Recovery in Cross QAM Communication Systems with the Reduced-Constellation Eighth-Order Estimator (RCEOE)

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Abstract—The eighth-order (EOE) phase estimator [4] is modified to work for an eight-symbol symmetrical constellation, so that the large signal-to-noise (SNR) performance is not limited by self-noise. By using only the eight highest energy points of cross-QAM constellations, a reduced constellation eighth-order estimator (RCEOE) is proposed. Computer simulations for 128-QAM show that this new method performs substantially better than the recently introduced APP phase estimator of Wang et al. [8]. However, simulations with 32-QAM show little performance advantage of the RCEOE over the APP estimator, for SNR values normally of interest, whereas for low SNR , the improvement is significant. Application to any constellation which can be reduced to an 8-symbol quadrant symmetrical sub-constellation is straightforward.

Index Terms—Synchronization, blind estimation, quadrature amplitude modulation, reduced constellation, carrier phase recovery, eighth-order estimator.

I. INTRODUCTION

THE need for blind phase recovery in quadrature amplitude modulation (QAM) systems is well established [1]-[9]. In order to satisfy this need, many systems have been invented. These systems can be grouped into two areas—those that require established gain control and those that do not. The fourth-power phase estimator [1]-[3], the eighth-order estimator (EOE) [4] and the Concentration Ellipse Orientation (CEO) estimator [9] are three systems in the latter category. Among the former category are the reduced-constellation fourth-power estimator [1], the two methods of Georghiades [1] which require finding the mode of the probability density of the phase, and more recently the optimal method, proposed by Wang and Serpedin [7], who along with Ciblat [8] have also introduced the APP estimator, which approximately implements the optimal estimator. For completeness, the hopelessly complex Minimum Distance Estimator (MDE) [5], and the Two-Stage Conjugate (2SC) algorithm, which is limited to square QAM systems in [5] and which according to Rice et al. [5] is similar to the Two-Pass algorithm of [6, pg. 33], are also noted in passing.

As shown by Wang et al. [8], the APP estimator works well for square QAM or low-level cross QAM constellations. However, there is still room for improvement for large cross QAM

constellations. The purpose of this paper is to introduce a method of significantly improving carrier phase recovery for such constellations. The method is similar to the APP and other estimators in that only part of the received constellation is utilized: indeed, APP utilizes only constellation points that lie on the diagonals so that the variance of the estimator is not limited by self-noise at large signal-to-noise (SNR) ratios. In contrast, our new method utilizes off diagonal points to accomplish the same goal—the advantage of this is that the highest energy points in cross QAM constellations can be used thereby increasing the likelihood that the variance of this new estimator will be smaller than any of the existing estimators, which use the lower energy diagonal points of cross QAM. Furthermore, even though this new estimator, called the reduced-constellation eighth-order estimator (RCEOE), uses eighth-order statistics, it is not simply the EOE of [4] used with the reduced-constellation. The EOE of [4] requires a suitable modification: this modification is not at all implicit in the work of [4], and hence will be described in Section IV.

This new estimator will be demonstrated for 128-QAM and 32-QAM. However, its application to other cross QAM systems, e.g. 512-QAM, is straightforward. Indeed, it can be applied to any constellation which is, or can be reduced to, a quadrant symmetrical 8-symbol constellation, such as the non-uniform 8-PSK of [10].

The rest of this paper is organized as follows. In Section II, the problem that is being solved is stated and addressed. This is followed by a review of the APP estimator in Section III, as the RCEOE will be compared to this. In Section IV, the eighth-order estimator is derived for a quadrant symmetrical eight symbol constellation. It is then shown in Section V how this new estimator can be applied to cross QAM systems. This is followed by Monte Carlo simulations in Section VI, which verify the usefulness of the RCEOE estimator. Finally, in Section VII, conclusions are drawn and future work is articulated.

II. STATEMENT OF THE PROBLEM

To describe the system we are interested in, we borrow from Georghiades [1], who assumes that the system is already equalized, frequency-synchronized, and that timing and relative gain control have already been achieved. With this being

the case, the baud-rate samples of the output of a matched filter are given by:

$$Y_n = X_n e^{j\theta} + V_n, \quad n = 0, 1, \dots, N-1 \quad (1)$$

where $X_n = X_{rn} + jX_{in}$ is a complex number that represents the M-QAM symbol transmitted at time nT , $1/T$ is the signaling rate, θ , which is assumed to be constant over the N symbols, is the unknown phase offset that is to be estimated, and V_n are complex independent identically distributed (i.i.d.) zero-mean Gaussian random variables with independent real and imaginary parts having variance σ^2 . The average constellation energy is assumed to be unity; hence, the symbol signal-to-noise ratio (SNR) is given by $SNR = 1/2\sigma^2$.

The received signal is given by $Y = Y_r + jY_i$, with

$$\begin{aligned} Y_r &= X_r \cos \theta - X_i \sin \theta + V_r, \\ Y_i &= X_r \sin \theta + X_i \cos \theta + V_i, \end{aligned} \quad (2)$$

where for notational convenience, explicit reference to n has been dropped.

The blind estimation problem is to find an estimate for θ , without actually detecting the data X . Note that because X has quadrant symmetry, it is only possible to recover θ within 90° .

III. REVIEW OF THE APP ESTIMATOR

The RCEO estimator will be compared to the APP estimator for 128-QAM and 32-QAM, because the APP estimator has the lowest variance of the practical estimators known until now [8]. Unfortunately, some pertinent details are missing from [8] for the 128-QAM case. Therefore, in this section, a brief review of the APP estimator will be given.

It will be convenient to rewrite (1) in polar form as

$$Y(n) = \rho(n) e^{j\phi(n)}. \quad (3)$$

The APP estimator [8] first transforms (3) to

$$Z(n) = F_{APP_M}(\rho(n)) e^{j4\phi(n)}, \quad (4a)$$

where $F_{APP_M}(\cdot)$ is a piecewise linear function that is unique to each M -QAM constellation. For example, for 128-QAM [11],

$$F_{APP_{128}}(\rho(n)) = \begin{cases} 492.9047\rho(n) & \text{if } \rho(n) \leq 0.24 \\ 1363.8\rho(n) + 33.5997 & \text{if } 0.42 \leq \rho(n) \leq 0.5 \\ 748.5407 & \text{if } 1.085 \leq \rho(n) \leq 1.095 \\ 293.3274 & \text{if } 1.095 < \rho(n) \leq 1.105 \\ 0 & \text{elsewhere.} \end{cases} \quad (4b)$$

On the other hand, for 32-QAM [8],

$$F_{APP_{32}}(\rho(n)) = \begin{cases} 206.9958\rho(n) & \text{if } \rho(n) \leq 0.5 \\ 608.4586\rho(n) + 2.2689 & \text{if } 0.84 \leq \rho(n) \leq 1.02 \\ 0 & \text{elsewhere.} \end{cases} \quad (4c)$$

The APP estimator then determines an estimate for the phase angle from

$$\hat{\theta} = \frac{1}{4} \text{angle} \left\{ - \sum_{n=0}^{N-1} Z(n) \right\}. \quad (5)$$

Note that N in (5) is the total number of received samples. As is evident from (4a), (4b) and (4c), some of the N received samples, i.e., the received off-diagonal constellation points, will contribute nothing to the sum in (5), because $F_{APP_M}(\cdot)$ is zero for those points. It is only the received diagonal constellation points which will add to the sum.

IV. EOE FOR A SYMMETRICAL 8-SYMBOL CONSTELLATION

As mentioned earlier, it is not self-evident from [4] how the EOE should be modified so that the large SNR performance is not limited by self-noise when used with the reduced constellation. (Self-noise refers to that part of the variance that is due solely to the received symbols). In order to describe the RCEO estimator, it is convenient to first show how the EOE estimator can be modified to work for a quadrant symmetrical eight-symbol constellation given by the points $(\pm k_1, \pm k_2)$ and $(\pm k_2, \pm k_1)$. Note that the constellation's points satisfy

$$(X_r^2 - k_1^2)(X_r^2 - k_2^2) + (X_i^2 - k_1^2)(X_i^2 - k_2^2) = 0, \quad (6)$$

Now suppose θ is known at the receiver. Then an estimate for the transmitted signal is given by

$$\begin{aligned} \hat{X}_r &= Y_r \cos \theta + Y_i \sin \theta \\ \hat{X}_i &= -Y_r \sin \theta + Y_i \cos \theta. \end{aligned} \quad (7)$$

If the effects of noise are ignored, i.e., as in the large SNR case, the estimated signal in (7) also satisfies (6). Substituting (7) into (6) and simplifying gives

$$A \cos 4\theta + B \sin 4\theta + C = 0, \quad (8)$$

where,

$$\begin{aligned} A &= (Y_r^4 + Y_i^4 - 6Y_r^2 Y_i^2) / 4, & B &= Y_r^3 Y_i - Y_r Y_i^3, \\ C &= \frac{3}{4} (Y_r^2 + Y_i^2)^2 - (k_1^2 + k_2^2)(Y_r^2 + Y_i^2) + 2k_1^2 k_2^2. \end{aligned}$$

However, θ is not known at the receiver. Nevertheless, an estimate for θ can be derived by noting that (8) is very similar to (4) of [4] (only the specific values of A , B , and C differ), which was solved for θ by minimizing the cost function

$$J = E[(A\alpha + B\beta + C)^2], \quad (9)$$

where α and β are parameters that are used to minimize J to arrive at:

$$\alpha = \frac{E(AB)E(BC) - E(B^2)E(AC)}{E(A^2)E(B^2) - E^2(AB)} \quad (10)$$

and

$$\beta = \frac{E(AB)E(AC) - E(A^2)E(BC)}{E(A^2)E(B^2) - E^2(AB)}. \quad (11)$$

As in [4], this minimization produces $\alpha = K \cos 4\theta$ and $\beta = K \sin 4\theta$, where K is a constant. Hence, it is straightforward to find an estimate for θ from

$$\hat{\theta} = \frac{1}{4} \tan^{-1} \left(\frac{\hat{\beta}}{\hat{\alpha}} \right), \quad (12)$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the estimates for (10) and (11); i.e., the expectations $E(\cdot)$ in (10) and (11) are evaluated with the sample mean $\frac{1}{N} \sum_{n=1}^N (\cdot)_n$.

Note that it is necessary to use the four quadrant inverse tangent function in (12).

From (8), (10) and (11), it is clear eighth-order statistics are being computed when the expected values are estimated. In practice, it is not necessary to compute the denominator in (10) and (11) as it cancels when substituted into (12).

In the *no noise* case, (8) is exact for each of the possible 8 symbols, i.e. the phase estimate which satisfies (8) does not change with the incoming data. This means that, in principle, the phase estimate can be estimated with zero variance, i.e., no self-noise. In fact, simulations show that even though the *estimates* for the numerators of (10) and (11) do change with the transmitted data, the phase estimate (12) is constant and not data dependent, provided the estimates for (10) and (11) are not both zero, which can happen for extremely small N values. (For example, for $N=1$, the estimates for (10) and (11) are always zero). Hence, (12) can be estimated without variance (self-noise) in the no-noise case, for practical values of N . Indeed, we will see that simulations show no leveling off of the estimator's variance as SNR increases, thereby providing evidence that self-noise is not limiting the performance of the RCEO estimator. This is unlike the case in [4] which is valid for more general constellations.

V. DESCRIPTION OF THE RCEO FOR CROSS-QAM SIGNALS

Recall that cross-QAM signals have eight outermost points that form an eight-point constellation as in Section IV above. For example, 32-QAM highest energy symbols are $(\pm 5, \pm 3)/\sqrt{20}$ and $(\pm 3, \pm 5)/\sqrt{20}$, those for 128-QAM are $(\pm 11, \pm 7)/\sqrt{82}$ and $(\pm 7, \pm 11)/\sqrt{82}$, and those for 512-QAM are $(\pm 23, \pm 15)/\sqrt{330}$ and $(\pm 15, \pm 23)/\sqrt{330}$, for constellations with unit energy.

Using (10), (11), and (12), the RCEO estimator forms an estimate of the phase based only on the received points that exceed a threshold, Th . This threshold is set halfway between the two outermost shells of the constellations. For 32-QAM, $Th = (\sqrt{26} + \sqrt{34})/\sqrt{80}$, whereas for 128-QAM,

$$Th = (\sqrt{170} + \sqrt{146})/\sqrt{328}, \text{ and}$$

$$Th = (\sqrt{754} + \sqrt{698})/\sqrt{1320} \text{ for 512-QAM.}$$

The implicit assumption is that the received points that exceed the threshold are from the outermost shell of the transmitted constellation, and hence phase estimation based on these are not limited by self-noise, as in the case of the eight-symbol symmetrical constellation, above. However, it is known that there is indeed a finite probability that the received symbols exceed the threshold even though the transmitted symbols do not. This probability increases with decreasing SNR . Neverthe-

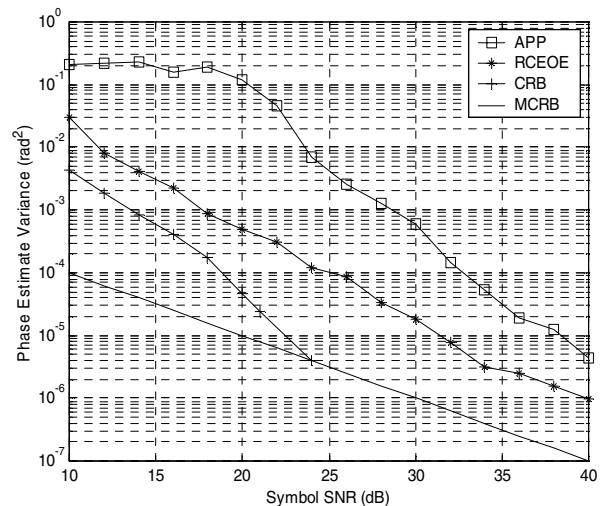


Fig. 1. Phase estimate variance for the RCEO and APP estimators with 128-QAM. $N=500$.

less, the simulations in the next section verify that this assumption is reasonable for the SNR of interest.

V. PERFORMANCE VERIFICATION

A. 128-QAM

In order to demonstrate the performance of the RCEO, Monte Carlo (MC) simulations were performed (assuming $\theta = 0.2$ radians) for 128-QAM with $N = 500$, and 1,000 trials (blocks) as in [8]. (Note: the number of trials is not to be confused with N ; indeed each trial involves N samples).

The results of these simulations are seen in Fig. 1, which also shows the performance of the APP estimator for 128-QAM, along with the performance given by the Cramer-Rao Bound (CRB) as determined by Rice et al. [5], and the simpler well-known Modified Cramer-Rao Bound (MCRB), which is equal to $(2N SNR)^{-1}$ and the CRB at high SNR [1].

As can be seen, the RCEO has substantially lower variance than the APP estimator, and unlike the latter, performs well even at low SNR ratios, which is important if coding is employed. Recall that for $SNR=27$ dB, the probability of symbol error for 128-QAM is approximately 10^{-3} , as shown in Fig. 2. Hence, if large coding gains are utilized, any method of phase estimation may have to work at SNR ratios substantially smaller than this, as the phase estimation is made prior to error correction.

It is also of particular interest to find the effects of the phase estimate on the probability of symbol error, P_{es} . This was done for 128-QAM using both the RCEO and APP estimators in Fig. 2, by using MC simulations to find the following expected value:

$$P_{es} \approx 0.5E \left[\begin{array}{l} \operatorname{erfc}\left((X_r - \hat{X}'_r + 1/\sqrt{82}) * \sqrt{SNR_a}\right) + \\ \operatorname{erfc}\left((-X_r + \hat{X}'_r + 1/\sqrt{82}) * \sqrt{SNR_a}\right) + \\ \operatorname{erfc}\left((X_i - \hat{X}'_i + 1/\sqrt{82}) * \sqrt{SNR_a}\right) + \\ \operatorname{erfc}\left((-X_i + \hat{X}'_i + 1/\sqrt{82}) * \sqrt{SNR_a}\right) \end{array} \right] \quad (13)$$

where

$$\begin{aligned} \hat{X}'_r &= X_r \cos(\theta - \hat{\theta}) - X_i \sin(\theta - \hat{\theta}) \\ \hat{X}'_i &= X_r \sin(\theta - \hat{\theta}) + X_i \cos(\theta - \hat{\theta}), \end{aligned}$$

and SNR_a is the SNR in absolute units, i.e. not in dB.

(This method of simulating P_{es} is called quasi-analytical estimation [12]. Strictly speaking, (13) is a very tight upper bound).

These results were obtained with a phase estimate determined after every block of $N = 500$ samples (symbols). Furthermore, 2,000 blocks were used: hence, a total of 1,000,000 symbols were utilized to obtain an estimate for P_{es} at every SNR value.

Notice that in the case of the system with no phase error, just Additive-White-Gaussian-Noise (AWGN),

$P_{es} \approx 2\operatorname{erfc}\sqrt{SNR_a}/82$. This curve is also plotted in Fig. 2. Notice also that the RCEOE estimator's performance shows very little increase in P_{es} over the no phase error system. On the other hand, the APP estimator's performance curves shows substantial degradation compared to the AWGN alone system. In fact, the RCEOE provides a gain of approximately 2.5 dB over the APP estimator at a symbol error rate of 10^{-5} and approximately 3 dB for 10^{-3} . Also, the APP curve shown in Fig. 2 is consistent with the one in [8].

It is also interesting to see how the variance changes with respect to the number of samples. From previous work [9], it is known that there is an inverse relationship between these two variables for the EOE estimator. That this is also the case for the RCEOE and APP estimators is verified in Fig. 3, which shows the variance as a function of N , for $SNR=25$ dB, whereas Fig. 4 shows the same for $SNR=30$ dB.

From Fig. 3 and Fig. 4, it is clear that for $N \geq 300$ there is no degradation in *improvement* of the RCEOE estimator's performance over that of the APP estimator's performance. That there is a minimum N required for the RCEOE estimator is not surprising: the estimator assumes that the transmitted constellation is symmetrical with eight symbols. Clearly, if N is too small, this assumption is violated. For example, if $N = 100$, the expected number of received symbols exceeding the threshold is only $8/128 * 100 = 6.25$. Hence, degradation in performance should be expected as the received constellation on which the RCEOE estimator operates (for each trial), is not likely to consist of eight symmetrical symbols. Indeed, there is a small but finite probability that in a given trial, there are no received symbols exceeding the threshold. When this happens, the RCEOE gives a phase estimate of zero. Thankfully, the probability of zero symbols decreases very rapidly with increasing

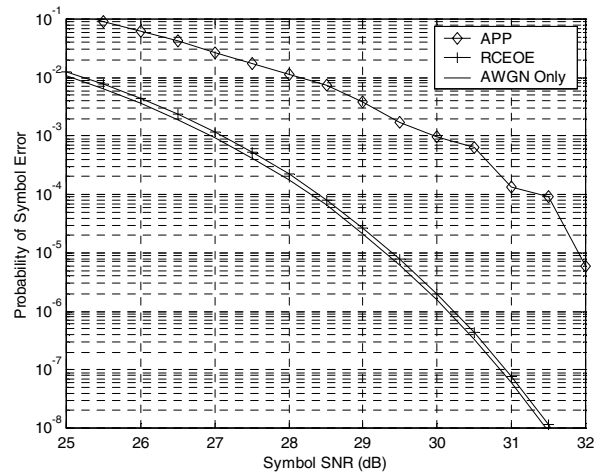


Fig. 2. Probability of symbol error for 128-QAM, $N = 500$.

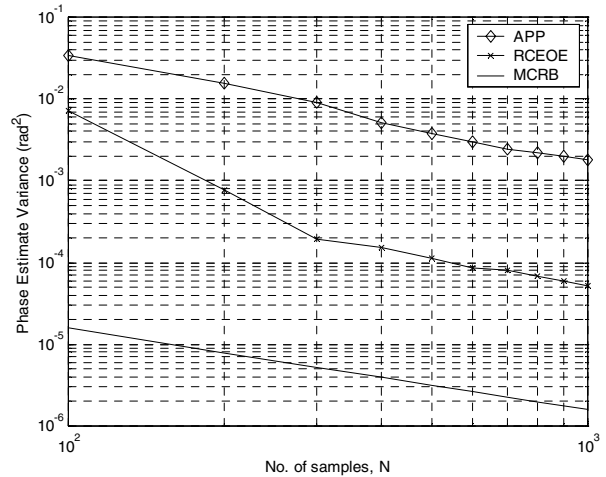


Fig. 3. Phase estimate variance as a function of the number of samples when $SNR = 25$ dB. For each N , the number of trials is 1000.

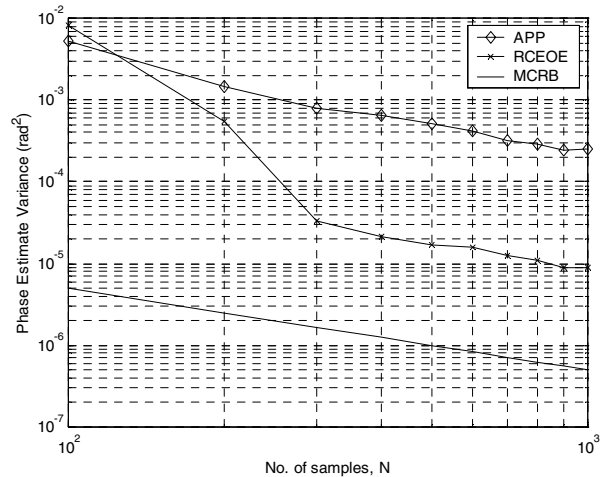


Fig. 4. Phase estimate variance as a function of the number of samples when $SNR = 30$ dB. For each N , the number of trials is 1000.

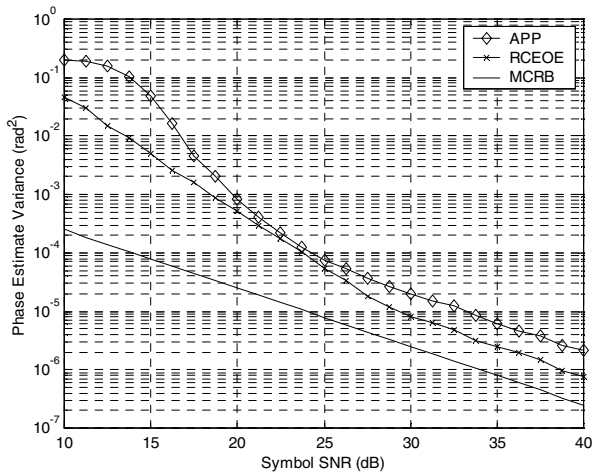


Fig. 5. Phase estimate variance for 32-QAM, N=200.

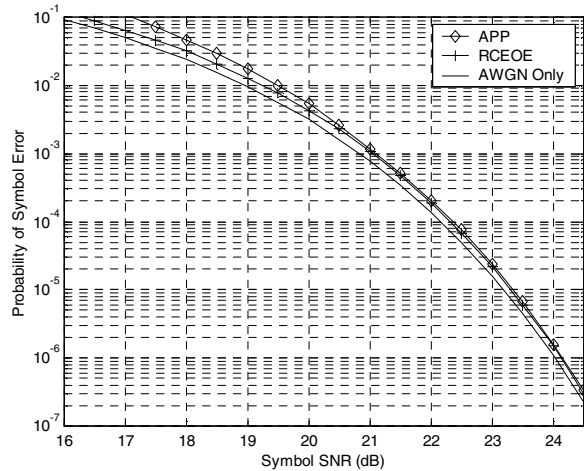


Fig. 6. Probability of symbol error for 32-QAM, N=200.

N. It should be noted that all reduced constellation estimators suffer from this, as is also noted in [1].

B. 32-QAM

So far, only 128-QAM has been discussed: yet, as mentioned earlier, our new estimator works for any level cross QAM constellation. However, simulations of the RCEOE estimator with 32-QAM, shown in Fig. 5 and Fig. 6, show that there is very little improvement over the APP estimator, for SNR values that are normally of interest. Nonetheless, if coding is employed, the substantial improvement (decrease in variance of the estimated phase) of the RCEOE over the APP estimator at low SNR, shown in Fig. 5, might be of some value. It will be left to future work to document the probability of error performance of coded 32-QAM signals with the RCEOE estimator, in comparison to that of the APP estimator.

Note that the probability of symbol error in Fig. 6 was calculated with (13), with the number 82 replaced by 20, and 2,000 blocks. The AWGN curve is given by $P_{es} \approx 2\text{erfc}\sqrt{\text{SNR}_a/20}$.

Fig. 6 also shows that for 32-QAM, there is very little degradation using $N=200$, as opposed to $N=500$, as in [8].

VII. CONCLUSIONS AND FUTURE WORK

A new phase estimator (RCEOE) has been introduced. The RCEOE performs substantially better than the best-known practical estimator, the APP, for 128-QAM. It has also been demonstrated that there is very little improvement of the RCEOE estimator over the APP estimator for 32-QAM, for SNR values normally of interest, whereas for low SNR, the improvement is significant, which might be of benefit to systems with large coding gains. Extension to other cross QAM constellations, e.g. 512-QAM, is straightforward. Indeed, our method can be applied to any constellation which is, or can be reduced to, a quadrant-symmetrical 8-symbol constellation.

Future work will investigate the viability of extending the Two-Stage Conjugate (2SC) algorithm [5] to cross QAM constellations, by using the APP and RCEOE estimators as its first stage. Additionally, the RCEOE described here only uses one level (i.e., 8 symbols of the same energy) of the constellation, whereas the APP estimator uses as many as it can. An investigation will be done to discover the viability of using more than one level with the RCEOE estimator.

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